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# Tensor categorical structure of observables in classical physics

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# Introduction of myself

- My name is Shogo Tanimura..
- I am a theoretical physicist.
- My main concerns are foundation of quantum theory, dynamical system theory, application of differential geometry to physics, and category theory.
- Today’s topic is a rather primitive application of category theory to both classical and quantum physics.

# Plan of this talk

1. Review of elementary dimensional analysis
2. Category of physical quantities
3. Dimensional analysis is formulated as tensor category
4. Unit system is a functor
5. Unit transformation law is a natural transformation
6. Observable algebra in quantum physics in terms of category

# Dimensional analysis

Calculus of physical quantities

3kg + 500g = 3000g + 500g = 3500g

4m + 70cm = 4m + 0.7m = 4.7m

40kg + 8cm = 48 ?   illegal !

Addition, equality and inequality of quantities must be homogeneous with respect to physical dimension.

# NONSENSE Sum

Fredrick I. Olness at Snowmass Village
http://www.physics.smu.edu//~olness/www/index.html

# Dimensional analysis in physics

[kg] = Mass

[meter] = Length

[sec] = Time

[pressure] = $[N/m^2] = MLT^{-2}L^{-2} = ML^{-1}T^{-2}$

[mass density] = $ML^{-3}$

$$\frac{[\text{pressure}]}{[\text{density}]} = L^2\, T^{-2}$$

$$[\text{velocity}] = L^1\, T^{-1} = \sqrt{\frac{[\textbf{pressure}]}{[\textbf{density}]}}$$

# Dimensional analysis in physics

Estimation of the sonic speed in air

Relevant parameters:

air pressure = 1 atm = $1.013 \times 10^5$ N/m$^2$

air density = 1.2kg/m$^3$

$$\frac{[\text{pressure}]}{[\text{density}]} = L^2\ T^{-2}$$

$$[\text{velocity}] = L^1\ T^{-1} = \sqrt{\frac{[\mathbf{pressure}]}{[\mathbf{density}]}} = 290 \text{m/s}$$

# Dimensional analysis in physics

Physicists feel uneasy with dimensionally incoherent formula :

$$E = mc^2 + \frac{1}{2}mv^2 + \underline{\frac{1}{4}mv^4}$$

Physicist notice this formula is wrong by a glance.

Correct formula:

$$E = mc^2\left\{1 + \frac{1}{2}\left(\frac{v}{c}\right)^2 + \frac{3}{8}\left(\frac{v}{c}\right)^4 + \cdots\right\} = \frac{mc^2}{\sqrt{1 - \left(\frac{v}{c}\right)^2}}$$

# Dimensional analysis in mathematics

Quadratic equation and its solution:

$$ax^2 + bx + c = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Requiring homogeneity of dimensions:

$$AX^2 = [ax^2] = [bx] = [c]$$

$$[a] = A, \qquad [b] = AX, \qquad [c] = AX^2$$

$$[x] = \frac{\left[-b \pm \sqrt{b^2 - 4ac}\right]}{[2a]} = \frac{AX}{A} = X$$

Consistent !

# Dimensional analysis in mathematics

Indefinite integral:

$$\int \frac{a}{a^2 + x^2} dx = \tan^{-1}\left(\frac{x}{a}\right)$$

Homogeneity of dimensions is kept.

# Dimensional analysis in quantum physics

Hamiltonian of harmonic oscillator:

$$\hat{H} = \frac{1}{2m}\hat{p}^2 + \frac{1}{2}m\omega^2\hat{x}^2 + mg\hat{x}$$

$$\hat{H}|\varphi_n\rangle = E_n|\varphi_n\rangle$$

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right) - \frac{mg^2}{2\omega^2}$$

$$\varphi_n(x) = \frac{1}{(2\pi)^{1/4}\Delta x^{1/2}} H_n\left(\frac{x}{\Delta x}\right) e^{-\frac{x^2}{4\Delta x^2}}$$

$$\Delta x^2 \coloneqq \frac{\hbar\omega}{2m\omega^2}$$

Homogeneity of dimensions is kept again.

# Question

Dimensional analysis works also in quantum theory.

However, the standard mathematical formulations of quantum theory (Hilbert space formalism, von Neumann algebra, C*-algebra) do not concern physical dimensions of observables.

I would like to formulate quantum theory in a language that involves physical dimensions.

# A Category of Simple Quantities

- Object: Simple quantity = 1-dim vector space
  - addition
  - scalar multiplication by real number
- Arrow: Linear mapping

Velocity $\mathrm{Hom}(L;T) = L \otimes T^*$

Time $T \to L$ Length, distance

$$t \longmapsto l = vt$$

$$v = 36\,\mathrm{km/h} = 10\mathrm{m}/s$$

# A Category of Simple Quantities

- Composition of linear mappings

Velocity $t \longmapsto l = vt$

$$\begin{tikzcd}
\text{Time} & T \arrow[r, "v"] \arrow[rd, "m \circ v"'] & L \arrow[d, "m"] & \text{Length, distance} \\
& & C & \text{Cost}
\end{tikzcd}$$

Meter $l \longmapsto c = ml$

$$v = 36\,\mathrm{km/h} = 10\mathrm{m/s}$$

$$m = 100\mathrm{yen}/300\mathrm{m} = \frac{1}{3}\mathrm{yen/m}$$

$$m \circ v = \frac{10}{3}\mathrm{yen/s} = 3.33\mathrm{yen/s}$$

# A Category of Simple Quantities

- Hom-set is also a quantity

$$V = \mathrm{Hom}(T; L) \ni v, \lambda v, v_1 + v_2$$

- Dual quantity: $T^* = \mathrm{Hom}(T; \mathbb{R})$
  - frequency, density
- Multilinear mappings define tensor products

$$V^* \otimes W^* \coloneqq \mathrm{Hom}(V, W; \mathbb{R}) \cong \mathrm{Hom}(V \otimes W; \mathbb{R})$$

Universality

$$\begin{tikzcd} V \times W \arrow[r, "\otimes"] \arrow[dr, "\forall f"'] & V \otimes W \arrow[d, "\exists! \hat{f}"] \\ & \forall Z \end{tikzcd}$$

# A Category of Simple Quantities

- The set of all endomorphisms of 1-dim vector space is isomorphic to real number field: dimensionless quantity

$$\mathrm{Hom}(L; L) \cong \mathrm{Hom}(T; T) \cong \mathbb{R}$$

- Unit and Coefficient:

$$V \cong \mathrm{Hom}(\mathbb{R}; V)$$

$$\boldsymbol{u} \longmapsto (E_{\boldsymbol{u}} : \lambda \longmapsto \lambda \boldsymbol{u})$$

$$\mathrm{kg} \longmapsto (E_{\mathrm{kg}} : 60 \longmapsto 60\mathrm{kg})$$

If $\boldsymbol{u} \neq \boldsymbol{0}$, $E_{\boldsymbol{u}}: \mathbb{R} \longrightarrow V$ is invertible:

$$\Theta_{\boldsymbol{u}} = (E_{\boldsymbol{u}})^{-1}: V \longrightarrow \mathbb{R}$$

# Higher-Dimensional Quantities

- Object: quantity = finite dimensional vector space over $\mathbb{R}$
- Arrow: linear mapping

- simple quantity: pressure

$$\text{area } L\otimes L \xrightarrow{p} F \text{ force}$$

- higher-dim quantity: tension

$$\dim S = \dim F = 3, \qquad \text{area } S\otimes S \xrightarrow{\tau} F \text{ force}$$

# Higher-Dimensional Quantities

- Multilinear mappings

$$V_1 \times V_2 \times \cdots \times V_k \longrightarrow W$$

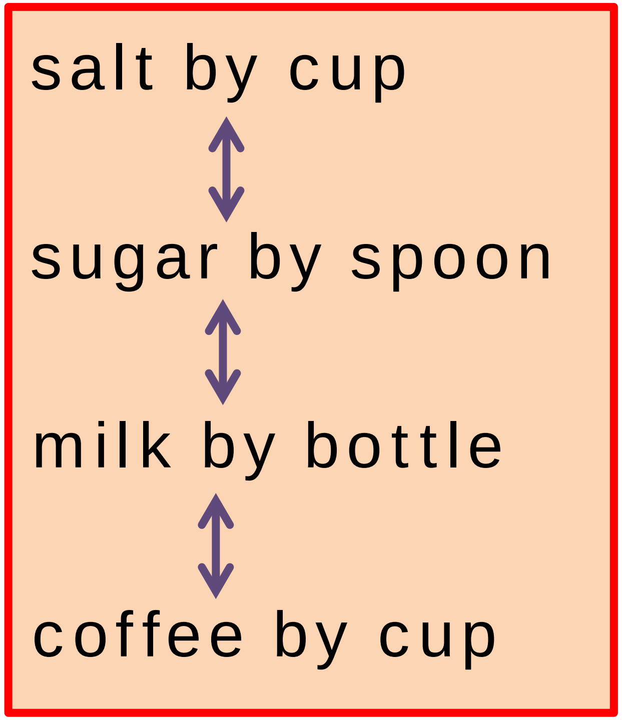
$$V \times V \times \cdots \times V \longrightarrow W$$

- symmetric bilinear ⇒ inner product
- antisymmetric ⇒ exterior product
- orientation ⇒ parity, twisted quantity
- contraction, trace

$$\begin{tikzcd}
V^* \times V \arrow[r, "\otimes"] \arrow[dr, "\text{pairing}"'] & V^* \otimes V \arrow[d, "\text{contraction}"] \\
& \mathbb{R}
\end{tikzcd}$$

# How to make equivalence classes of quantities?

When we have transformation laws among measurable quatitities but no general measure



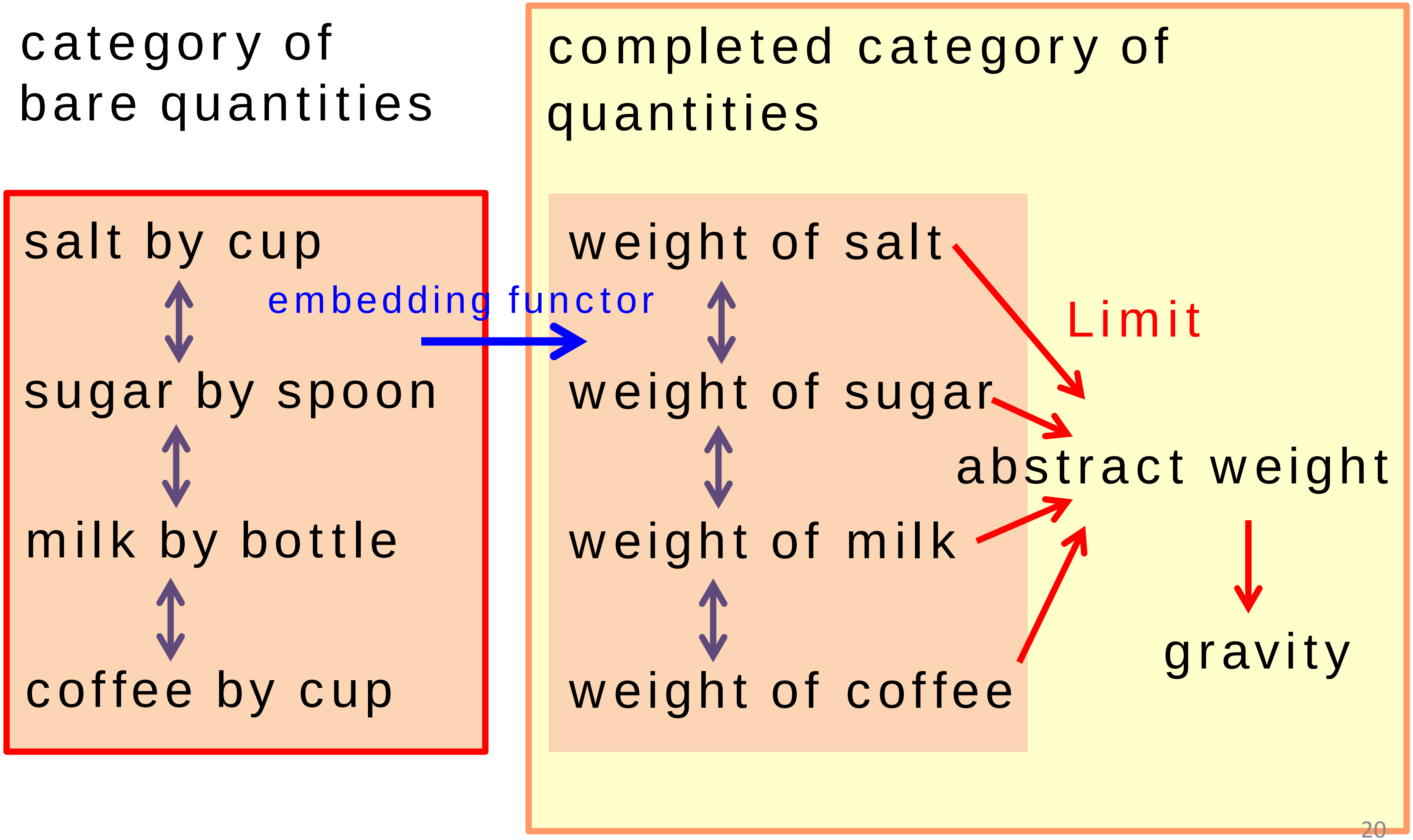
salt by cup

sugar by spoon

milk by bottle

coffee by cup

# Abstractization of Quantities



category of bare quantities

salt by cup

sugar by spoon

milk by bottle

coffee by cup

embedding functor

completed category of quantities

weight of salt

weight of sugar

weight of milk

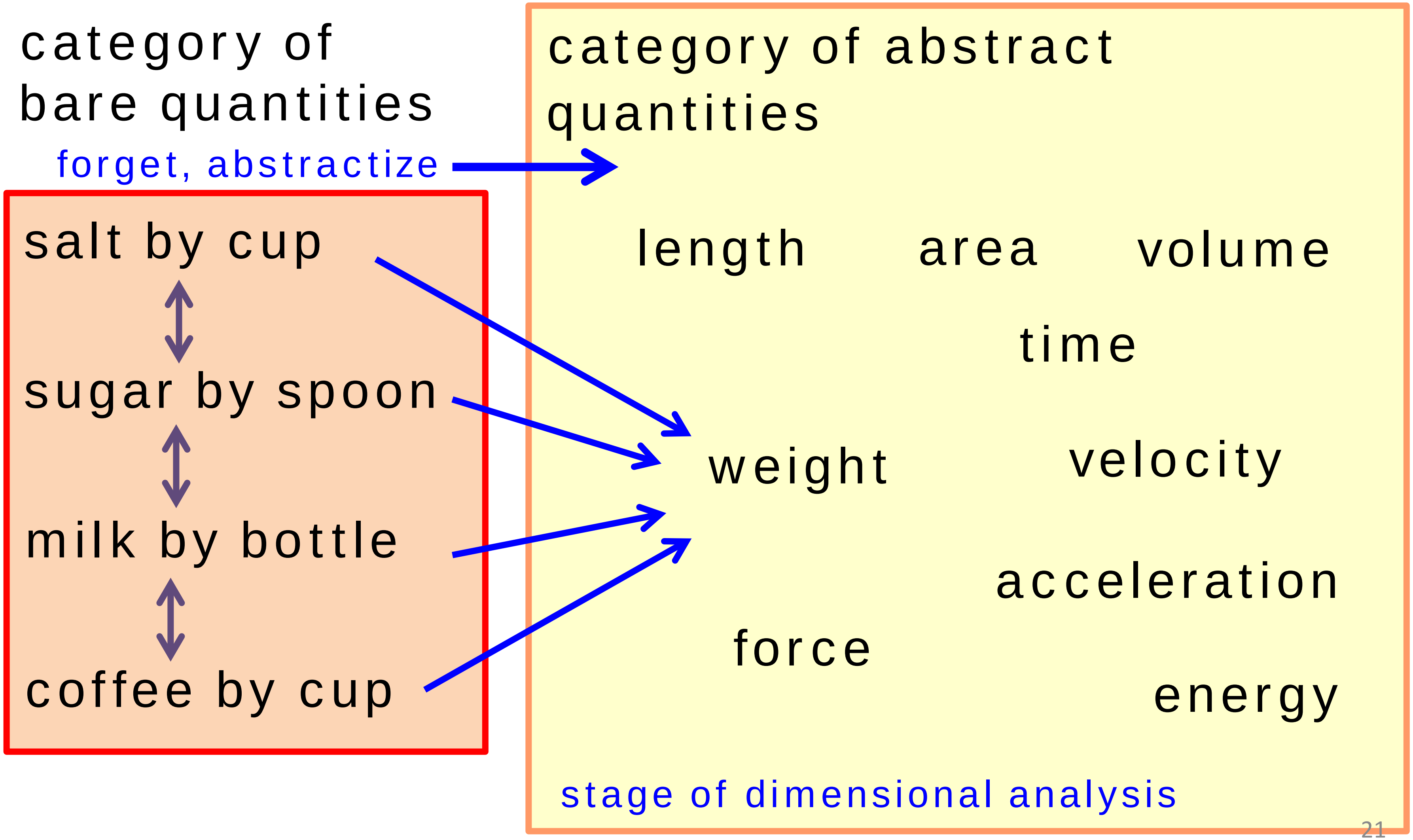
weight of coffee

Limit

abstract weight

gravity

# Abstractization of Quantities



category of bare quantities

forget, abstractize

salt by cup

sugar by spoon

milk by bottle

coffee by cup

category of abstract quantities

length

area

volume

time

weight

velocity

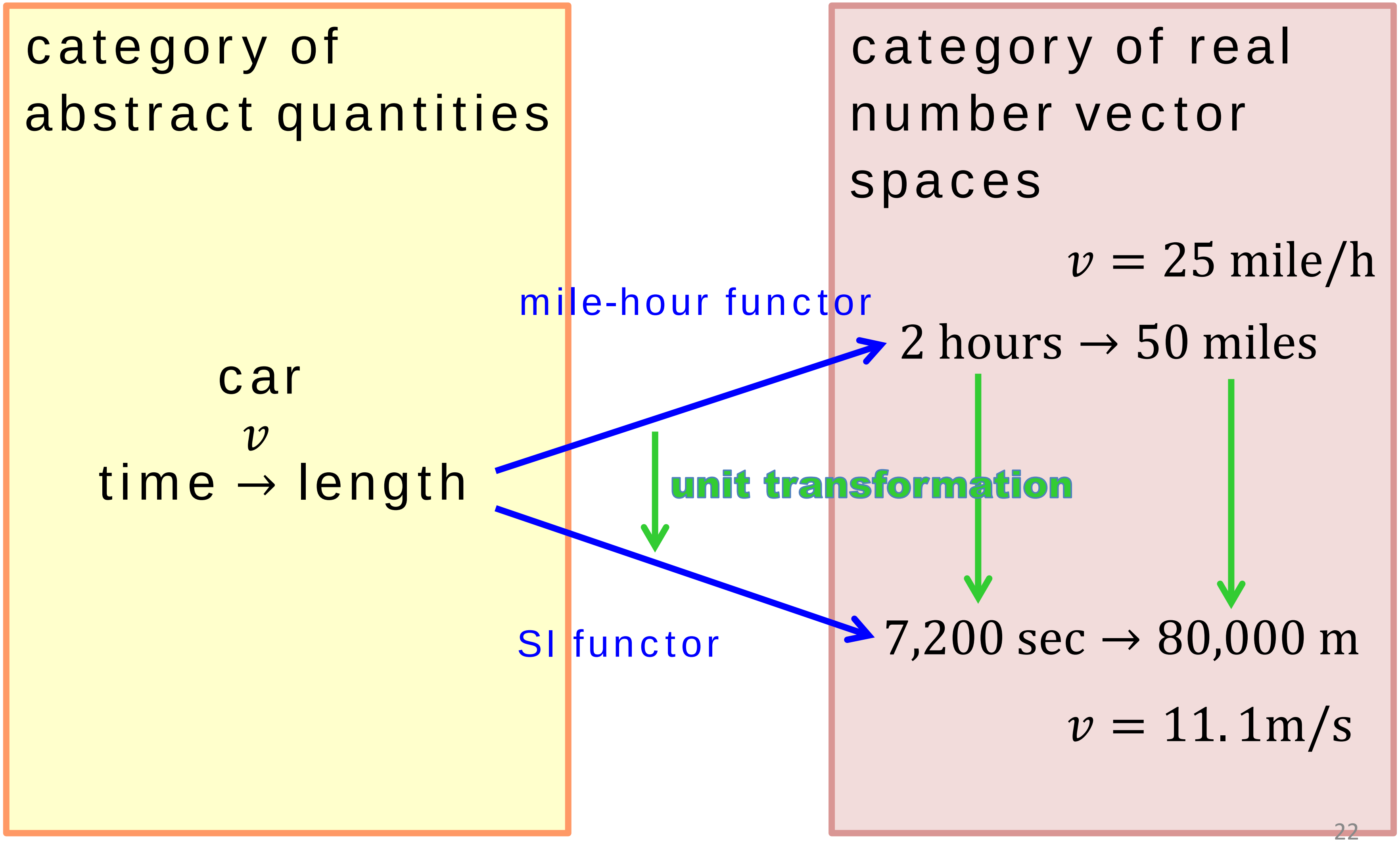
acceleration

force

energy

stage of dimensional analysis

# Unit system is a functor,
# Unit transformation is a natural transformation



category of abstract quantities

car

$v$

time → length

mile-hour functor

SI functor

**unit transformation**

category of real number vector spaces

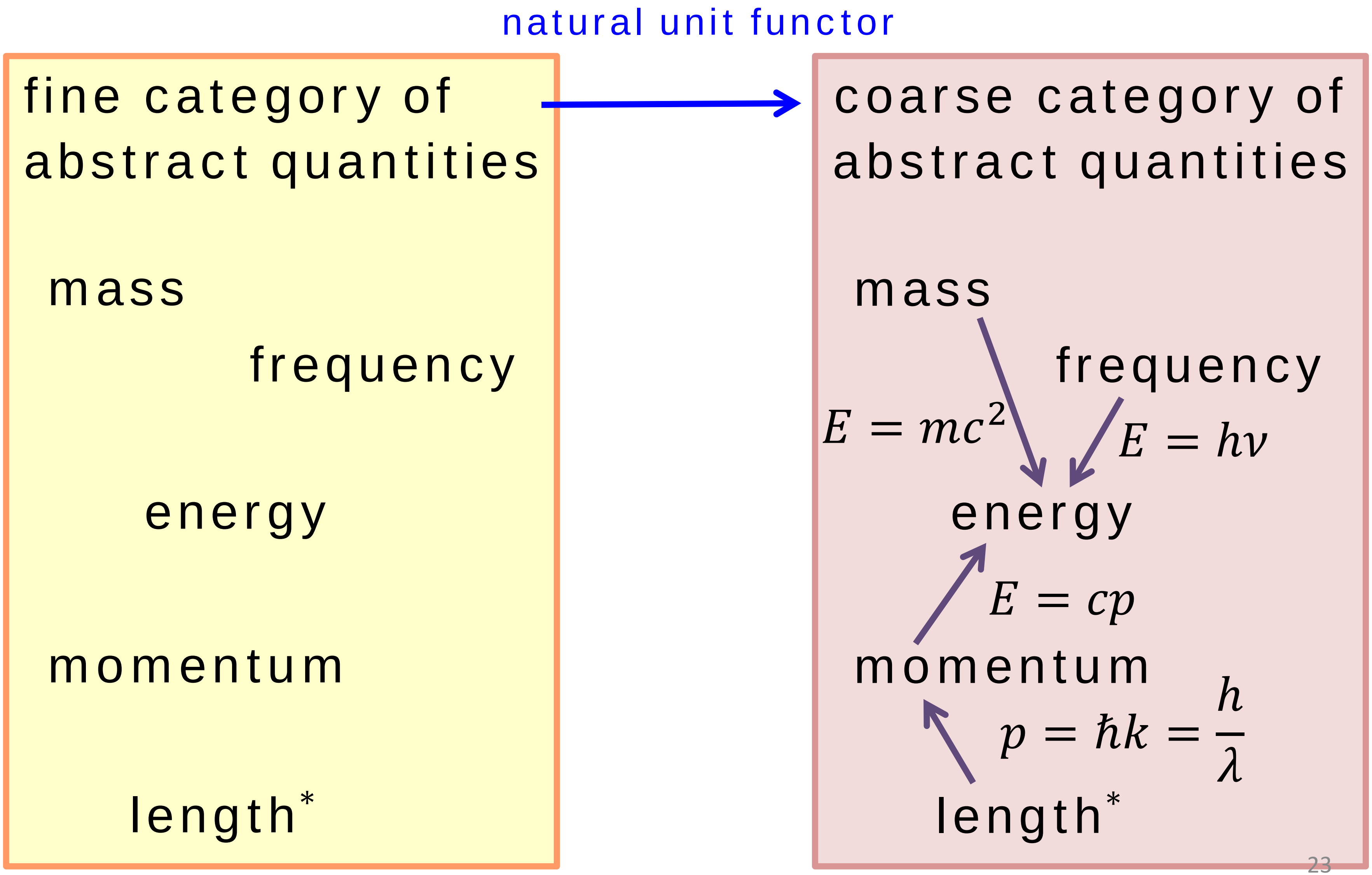
$v = 25\ \mathrm{mile/h}$

$2\ \mathrm{hours} \rightarrow 50\ \mathrm{miles}$

$7{,}200\ \mathrm{sec} \rightarrow 80{,}000\ \mathrm{m}$

$v = 11.1\mathrm{m/s}$

# Fineness of classification



natural unit functor

fine category of abstract quantities

mass

frequency

energy

momentum

length*

coarse category of abstract quantities

mass

frequency

$E = mc^2$

$E = h\nu$

energy

$E = cp$

momentum

$p = \hbar k = \frac{h}{\lambda}$

length*

# How to quantize?
# = How to make non-commutative?

- Tensor product of classical quantities is symmetric:

$$X \otimes Y \cong Y \otimes X$$

- In quantum theory, we would like to introduce non-commutative products of observables.
- Algebraic structure must be consistent with dimensional analysis:

$$E = \frac{1}{2m} p^2 + \frac{1}{2} m\omega^2 x^2 + mgx$$

# Toward quantization

category of classical physical quantities

$1 \qquad T \qquad M$

$L \qquad L^{*} = L^{-1}$

$L \otimes L \qquad M \otimes L$

$P = M \otimes L \otimes T^{-1}$

$E = M \otimes L^{2} \otimes T^{-2}$

category of quantum observables

$* \xrightarrow{1} *$

$* \xrightarrow{x} L *$

$* \xrightarrow{y} L *$

$* \xrightarrow{p_x} P *$

$* \xrightarrow{x+y} L *$

$\mathrm{Hom}(A *, B *)$ is required to be a vector space.

# Category of quantum observables

object: asterisk marked by classical quantity

arrow: quantum observable

Each hom-set is a vector space over $\mathbb{C}$.

composition: product of quantum observables

$$\begin{tikzcd}
* \arrow[r, "x"] \arrow[dr, "p_x \circ x"'] & L* \arrow[d, "p_x"] \\
 & P \otimes L*
\end{tikzcd}
\qquad
\begin{tikzcd}
* \arrow[r, "x"] \arrow[d, "p_x"'] & L* \arrow[d, "p_x"] \\
P* \arrow[r, "x"'] & P \otimes L*
\end{tikzcd}$$

$$p_x \circ x \neq x \circ p_x$$

# Algebraic equation of quantum observables

Description of harmonic oscillator hamiltonian

$$\begin{tikzcd}
{*} \arrow[r, "x"] \arrow[dr, "p_x"'] & L\,{*} \arrow[r, "x"] & L^2\,{*} \arrow[r, "\frac{1}{2}m\omega^2"] & M\otimes L^2\otimes T^{-2}\,{*} \arrow[d, equal] \\
 & P\,{*} \arrow[r, "p_x"'] & P^2\,{*} \arrow[r, "\frac{1}{2m}"'] & M^{-1}\otimes P^2\,{*}
\end{tikzcd}$$

$$* \xrightarrow{H = \frac{1}{2m}p_x^2 + \frac{1}{2}m\omega^2 x^2} M\otimes L^2\otimes T^{-2} \; *$$

# Representation functor sends objects to Hilbert spaces and observables to operators

Category of Hilbert spaces with physical-quantity coefficients

$$\begin{tikzcd}
\mathfrak{H} \arrow[r, "\hat{x}"] \arrow[dr, swap, "\widehat{p_x}"] & L\otimes\mathfrak{H} \arrow[r, "\hat{x}"] & L^2\otimes\mathfrak{H} \arrow[r, "\frac{1}{2}m\omega^2\hat{1}"] & M\otimes L^2\otimes T^{-2}\otimes\mathfrak{H} \arrow[d, equal] \\
 & P\otimes\mathfrak{H} \arrow[r, swap, "\widehat{p_x}"] & P^2\otimes\mathfrak{H} \arrow[r, swap, "\frac{1}{2m}\hat{1}"] & M^{-1}\otimes P^2\otimes\mathfrak{H}
\end{tikzcd}$$

$$\mathfrak{H} \xrightarrow{\hat{H} = \frac{1}{2m}\hat{p}_x^2 + \frac{1}{2}m\omega^2\hat{x}^2} M\otimes L^2\otimes T^{-2}\otimes\mathfrak{H}$$

# Eigenvector subspace is an equalizer

Category of Hilbert spaces with physical-quantity coefficients.

Eigenvalue problem: $\widehat{H}|\varphi\rangle = \varepsilon|\varphi\rangle$

$$\begin{tikzcd}
\mathcal{L} \arrow[r] & \mathfrak{H} \arrow[r, shift left, "\widehat{H}"] \arrow[r, shift right, "\varepsilon\hat{1}"'] & E\otimes\mathfrak{H}
\end{tikzcd}$$

# Summary 1/3

1. Quantities in classical physics forms a tensor category.
2. A set of isomophic quantities define an abstract quantity.
3. Functor sends a fine category to a coarse category of quantities.
4. Unit system is a functor from quantities to real values.
5. Unit transformation law is a natural transformation.

# Summary 2/3

6. An observable in quantum physics is an arrow that multiplies physical quantity on an object:

$$* \xrightarrow{x} L * \qquad\qquad M * \xrightarrow{x} L \otimes M *$$

7. Hom-set is a vector space over $\mathbb{C}$.
8. This structure guarantees homogeneity of physical dimensions of terms in equations of observables.

# Summary 3/3

9. Composition of quantum observables is non-commutative in general.
10. Representation functor sends objects to Hilbert spaces with physical-quantity coefficients and arrows to operators:

$$\mathfrak{H} \xrightarrow{\hat{x}} L\otimes\mathfrak{H} \qquad\qquad M\mathfrak{H} \xrightarrow{\hat{x}} L\otimes M\otimes\mathfrak{H}$$

11. Spectral values of observable operators have suitable physical dimensions.

# References

1. Tanimura, “Topology, category, and differential geometry: from the viewpoint of duality” (in Japanese)
2. Tanimura, Series of lectures on geometry and physics published in Mathematical Science (in Japanese)
3. Kitano, “Mathematical structure of unit systems” J. Math. Phys. 54, 052901 (2013)

# Thank you for your attention