Quantum Information Processing in Non-Markovian Quantum Complex Systems
2019, December

Geometric interpretation of weak value in quantum theory and geometric picture for negative probability

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I am

- a professor of Nagoya University
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- investigating fundamental aspects of quantum theory

This talk

1. Introduction to the notion of weak value — formalism
2. Experimental demonstration
3. Geometric interpretation of weak value
4. Implication

Weak value

- In 1988, Aharonov, Albert, Vaidman proposed a notion of weak value.
- It is named "weak value" since it is measured via "weak measurement", in which interaction between an object and an apparatus is weak.
- It is more sensible to call an expectation value conditioned by a set of an initial state and a final state.

Yakir Aharonov

https://history.aip.org/phn/11408012.html

Difference of observable and value

Observables

- admitting algebraic manipulations like sum, product, and scalar multiplication.
- Products are noncommutative in general.

- $L_1 + L_2 = L_3$ (sum of lengths)
- $L = 2\pi r, \;\; S = \pi r^2, \;\; S = \frac{1}{2}ab$
- $E = mc^2, \;\; E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 + mgz$
- $H = \frac{1}{2m}p^2 + \frac{1}{2}kq^2, \quad qp - pq = i\hbar$
- $S_n = S_x + S_y$

Difference of observable and value

- Value

  Measurement of an observable yields a real number if a unit quantity is defined.
- $\varepsilon(M) = 55\ \mathrm{kg}$ (ε : evaluation map)
- It is allowed to write $M = 55\ \mathrm{kg}$
- It is better to write $M \to 55\ \mathrm{kg}$ or $M \leftarrow 55\ \mathrm{kg}$
- $\varepsilon(L) = 1.67\ \mathrm{m}$

Subtleness of quantum observables

In general,

Value of (sum of observables)
$\neq$ Sum of (values of observables)

For example,

- $H = \frac{1}{2m}p^2 + \frac{1}{2}kq^2$
  - Measurement of p yields continuous $-\infty \leq p \leq \infty$
  - Measurement of q yields continuous $-\infty \leq q \leq \infty$
  - However, measurement of H yields disctere $\hbar\omega\left(n + \frac{1}{2}\right)$
- $S_n = S_x + S_y$
  - Individual measurements of S_x, S_y yield ± 1
  - However, measurement of S_n yields $\pm\sqrt{2}$

Subtleness of quantum observables

In general,

Value of (sum of observables)

$\neq$ Sum of (values of observables)

Namely,

there is no nontrivial homomorphism from the noncommutative algebra of observables to the commutative algebra of real numbers.

Three kinds of values in quantum theory

1. Eigenvalue (spectral value)

$$\hat{A}|\varphi_i\rangle = a_i|\varphi_i\rangle$$

- $\{a_1, a_2, a_3, \cdots\}$
- Yield of individual measurement

2. Expectation value

$$\langle\hat{A}\rangle = \mathbf{E}\left[\hat{A}\right] = \langle\psi|\hat{A}|\psi\rangle$$

- Average of accumulated data

3. Weak value

$$w(\hat{A}) = \frac{\langle\psi_{\mathrm{fin}}|\hat{A}|\psi_{\mathrm{ini}}\rangle}{\langle\psi_{\mathrm{fin}}|\psi_{\mathrm{ini}}\rangle}$$

Properties of weak value

Weak value

$$w(\hat{A}) = \frac{\langle \psi_{\mathrm{fin}} | \hat{A} | \psi_{\mathrm{ini}} \rangle}{\langle \psi_{\mathrm{fin}} | \psi_{\mathrm{ini}} \rangle}$$

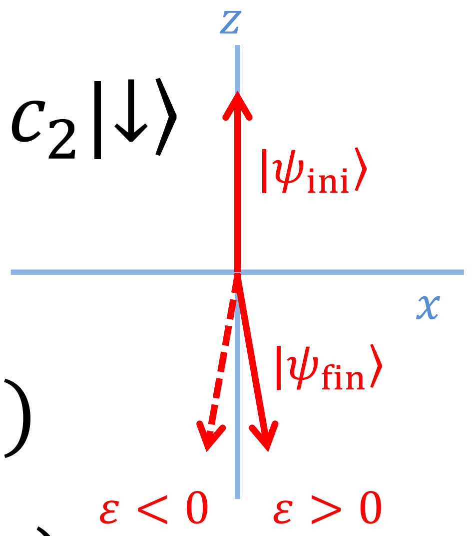
1. Complex number
2. Invariant under arbitrary phase transformation $|\psi_k\rangle \to e^{i\theta_k}|\psi_k\rangle$
3. Even if $\hat{A}$ has the maximum eigenvalue a_{max} and the minimum eigenvalue a_{min},

$$a_{\mathrm{min}} \le \mathbf{Re}\, w(\hat{A}) \le a_{\mathrm{max}}$$

does NOT hold in general.

Example of calculation of weak value

spin $\frac{1}{2}$ (2-state system) $|\psi\rangle = c_1|\uparrow\rangle + c_2|\downarrow\rangle$



$|\psi_{\text{ini}}\rangle = |\uparrow\rangle$

$|\psi_{\text{fin}}\rangle = \varepsilon|\uparrow\rangle + \sqrt{1-\varepsilon^2}|\downarrow\rangle \quad (|\varepsilon| \ll 1)$

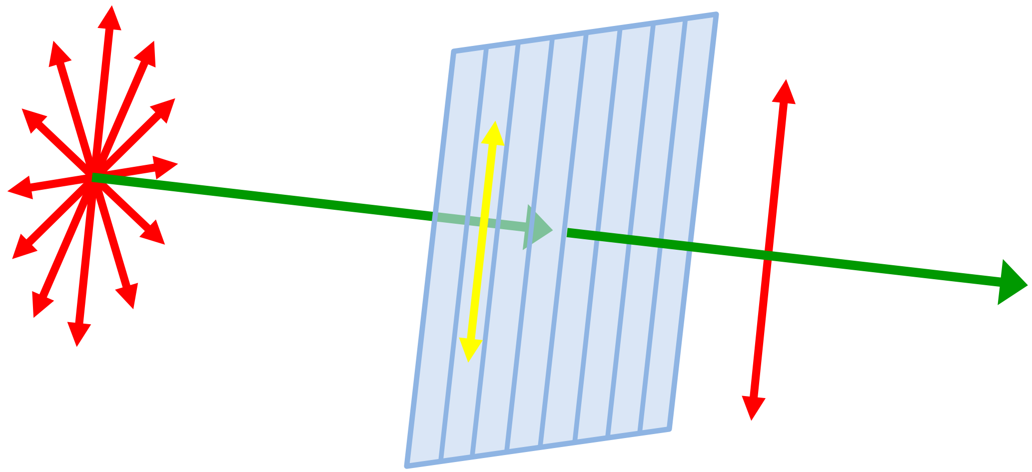
$$\hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$w(\hat{\sigma}_x) = \frac{\langle\psi_{\text{fin}}|\hat{\sigma}_x|\psi_{\text{ini}}\rangle}{\langle\psi_{\text{fin}}|\psi_{\text{ini}}\rangle} = \frac{\sqrt{1-\varepsilon^2}}{\varepsilon} \to \pm\infty$$

$(\varepsilon \to \pm 0)$

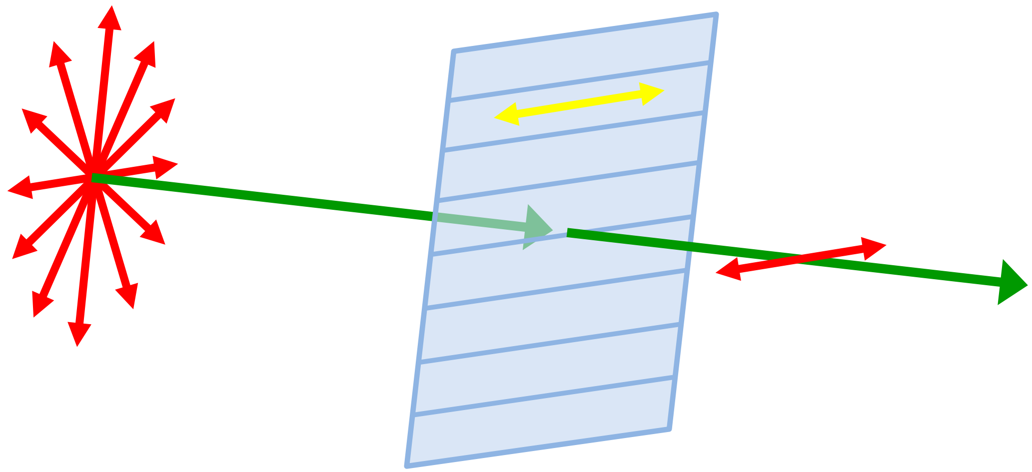
Experiments

Filter for polarization of light



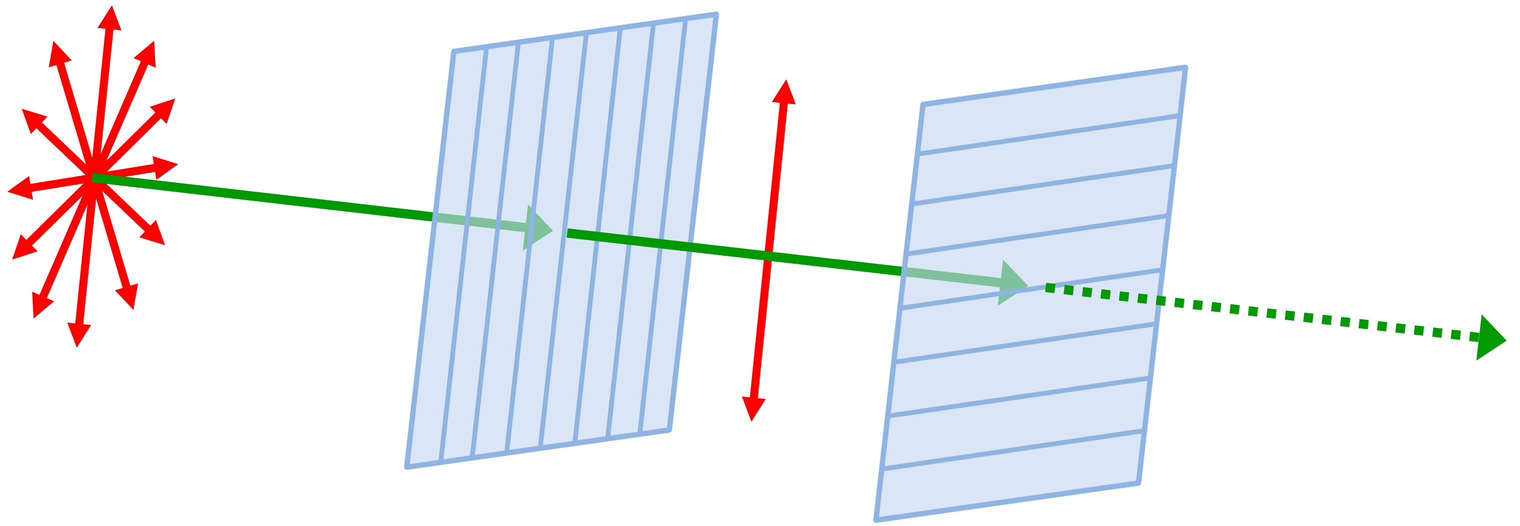
Polarization filter

selectively permits pass of light



Orthogonal filters

do not permit light pass at all



Polarization filters

Orthogonal filters

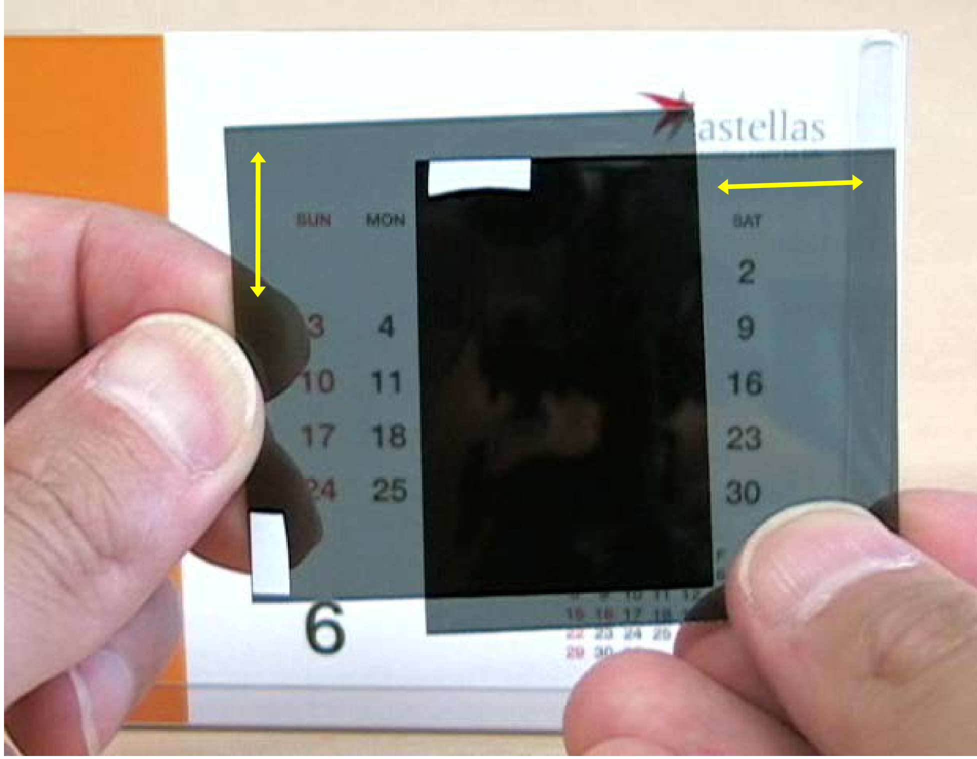
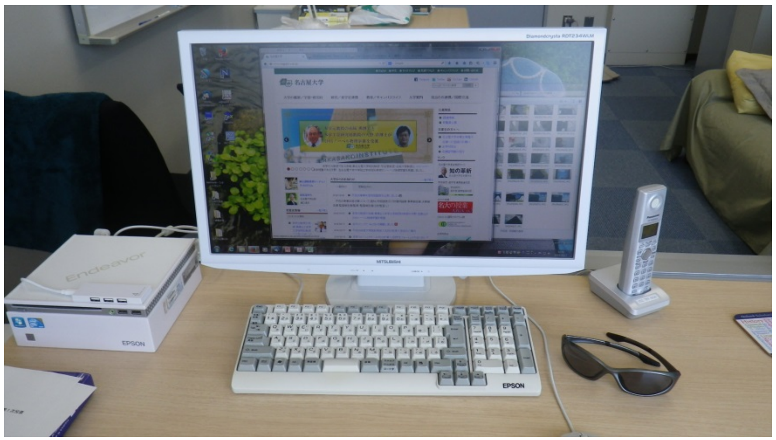
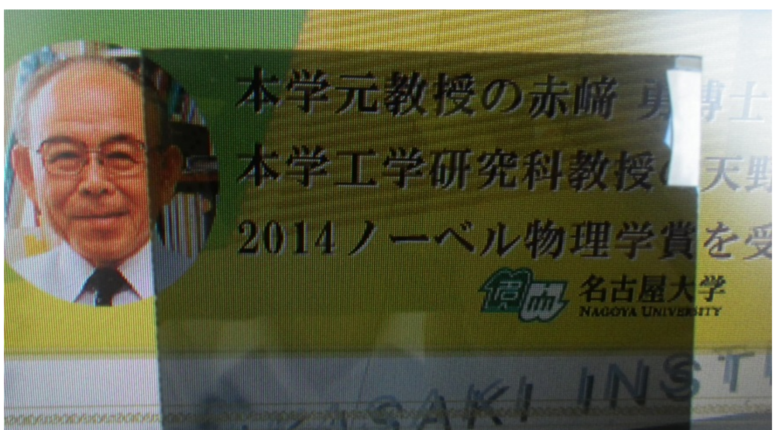


Photo by Tanimura

Application of polarization

Liquid crystal display



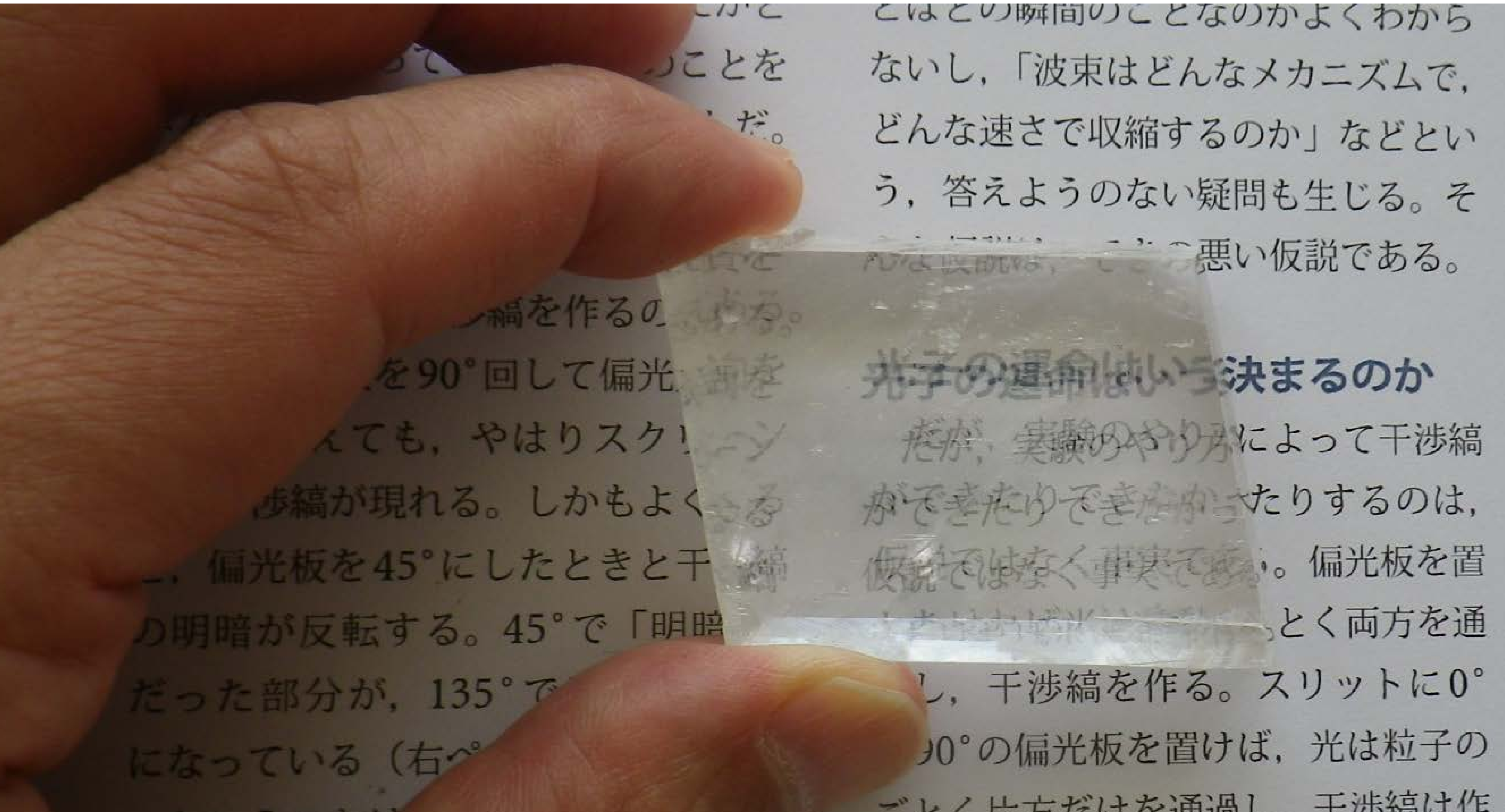


Application of polarization

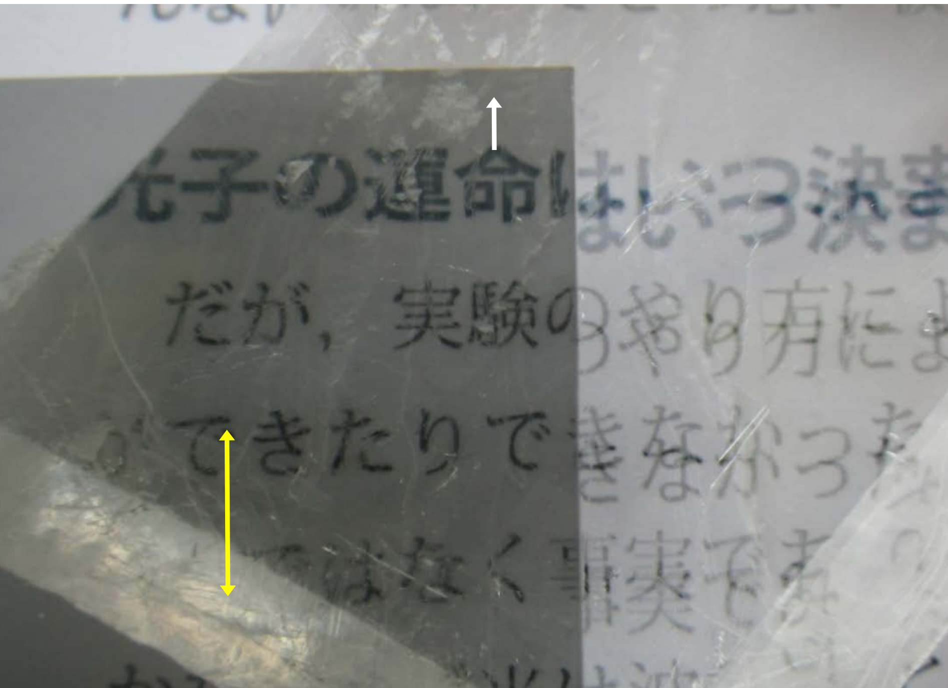
Sunglasses

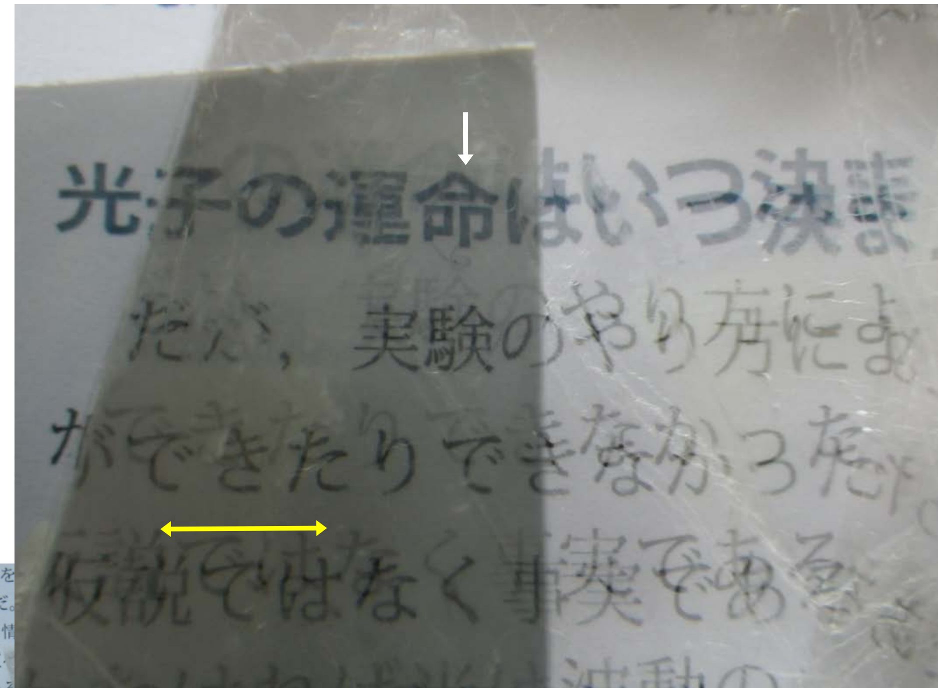
Birefringence

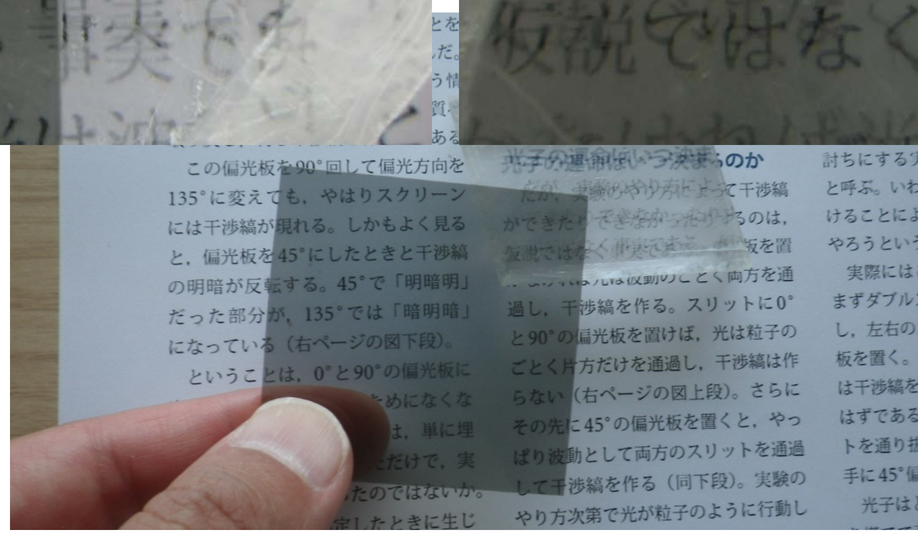
Calcite



Polarization filter on birefringence

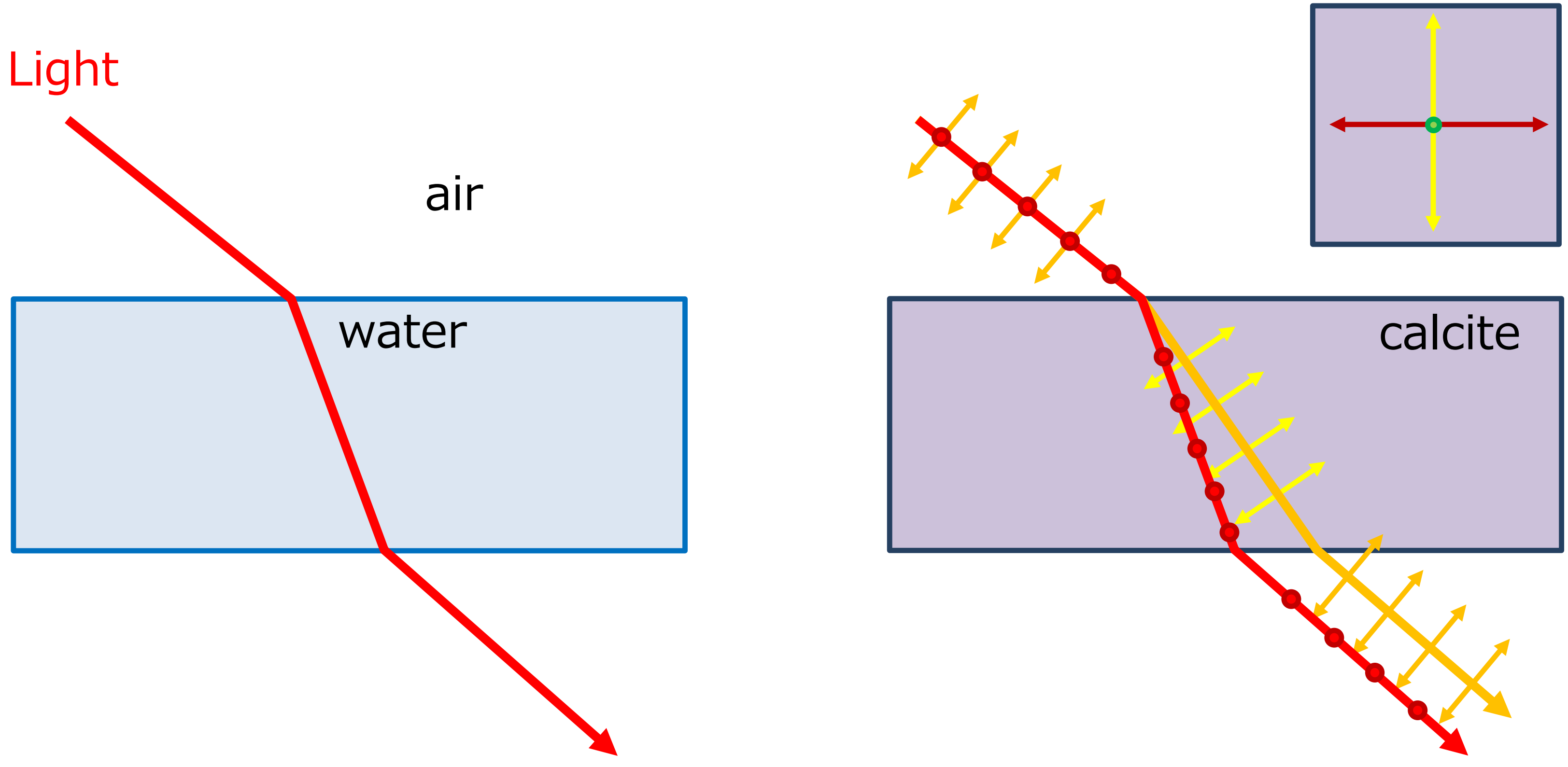






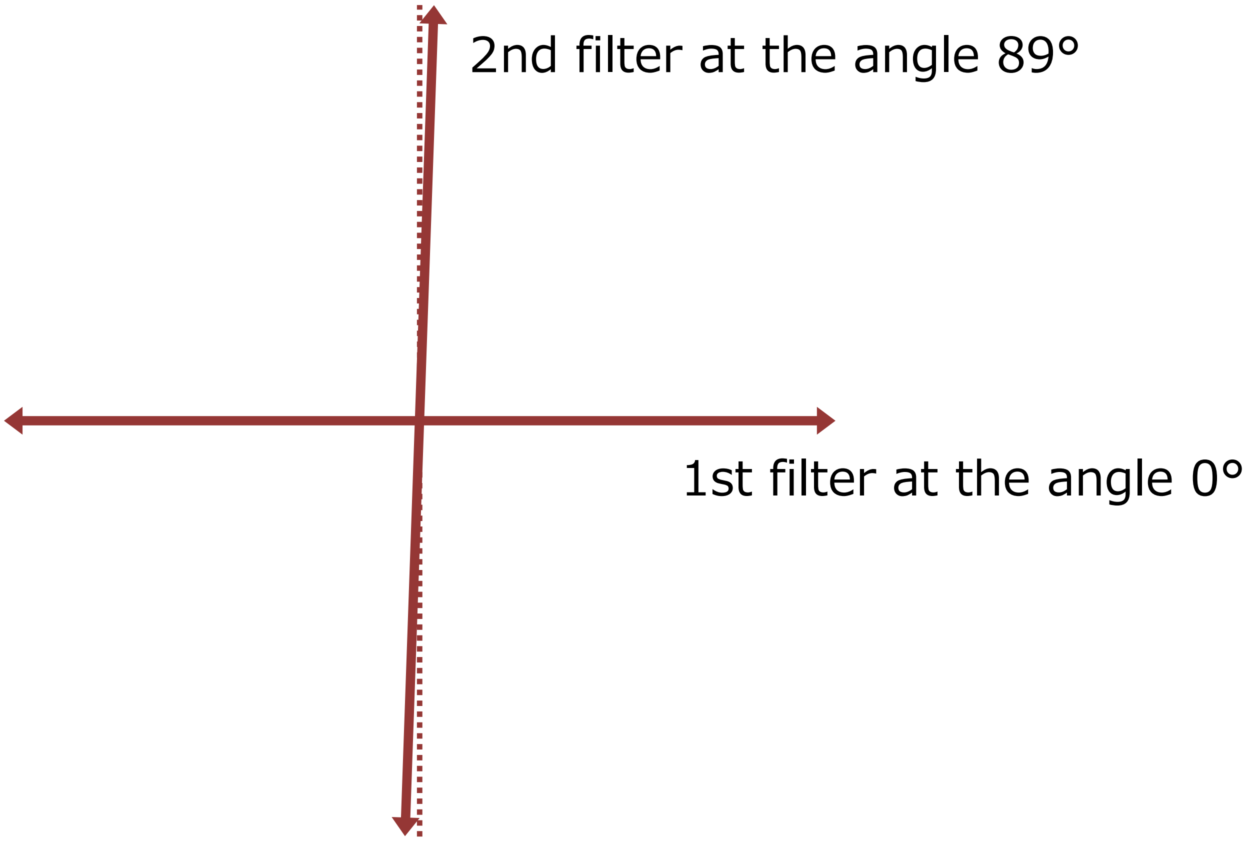
Birefringence

Birefringence decomposes light into two orthogonal polarization components.



Experimental realization of weak value 1/3

Almost orthogonal polarization filters

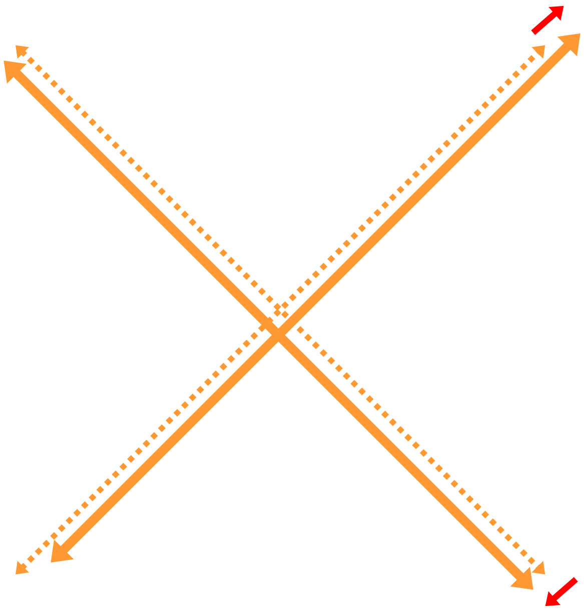


theory : Duck, Stevenson, Sudarshan: Phys. Rev. D (1989)
experiment : Ritchie, Story, G. Hulet: Phys. Rev. Lett. (1991)

Experimental realization of weak value 2/3

Aliment of the birefringent crystal

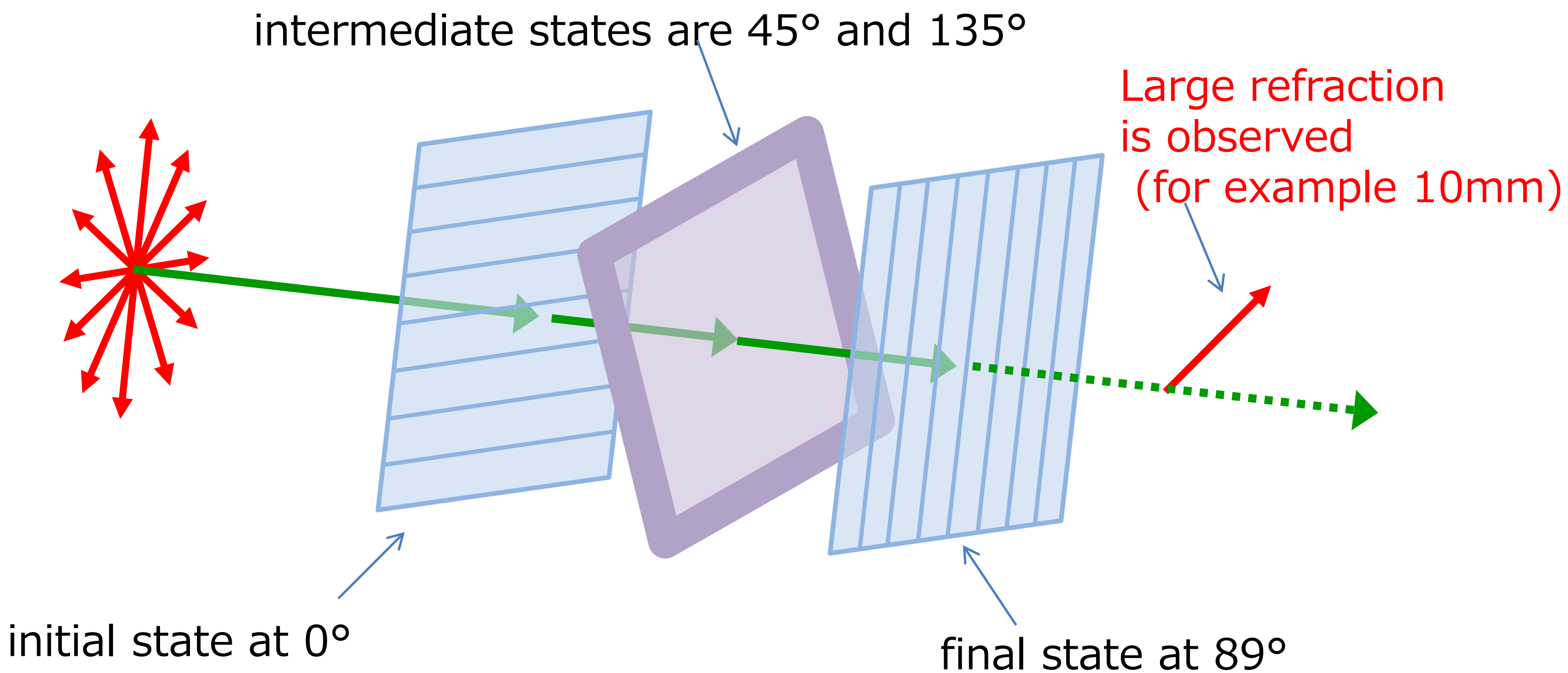
45°-polarized light is refracted by 1mm



135°-polarized light is refracted by 1mm
in the opposite direction

Experimental realization of weak value 3/3

Insert the birefringent crystal into the two almost orthogonal filters



Expectation value with nonnegative probability

Theorem: If $\hat{A}$ has the maximum eigenvalue $a_{\max}$ and the minimum eigenvalue $a_{\min}$, it holds that

$$a_{\min} \le \langle \hat{A} \rangle \le a_{\max}$$

Proof: By assumption

$$a_{\min} \le a_i \le a_{\max}$$

For probability p_i satisfying $0 \le p_i \le 1$, $\sum_i p_i = 1$,

$$p_i a_{\min} \le p_i a_i \le p_i a_{\max}$$

$$a_{\min} = \sum_i p_i a_{\min} \le \sum_i p_i a_i \le \sum_i p_i a_{\max} = a_{\max}$$

Probabilistic interpretation of weak value

$$w(\hat{A}) = \frac{\langle \psi_{\mathrm{fin}} | \hat{A} | \psi_{\mathrm{ini}} \rangle}{\langle \psi_{\mathrm{fin}} | \psi_{\mathrm{ini}} \rangle} = \sum_i p_i a_i$$

If $a_{\min} \le w(\hat{A}) \le a_{\max}$ does not hold, we must discard at least one of the two assumptions, $0 \le p_i \le 1$ or $\sum_i p_i = 1$. On the other hand, the normalization condition

$$w(\hat{1}) = \frac{\langle \psi_{\mathrm{fin}} | \hat{1} | \psi_{\mathrm{ini}} \rangle}{\langle \psi_{\mathrm{fin}} | \psi_{\mathrm{ini}} \rangle} = 1 = \sum_i p_i$$

holds. Therefore, we must discard $0 \le p_i \le 1$.

Model of weak measurement

object: $|\psi\rangle \in \mathfrak{H}$ $\quad \hat{A} = \sum_a a\hat{\Pi}_a \quad \hat{B} = \sum_b b\hat{\Pi}_b$

apparatus: $|\lambda\rangle \in \mathcal{L}$, $\quad \hat{M}$ (meter observable)

composite system: $|\psi\rangle \otimes |\lambda\rangle \in \mathfrak{H} \otimes \mathcal{L}$

interaction: $|\psi\rangle \otimes |\lambda\rangle \mapsto \hat{U}_g |\psi\rangle \otimes |\lambda\rangle$ (g:coupling constant)

We want to know the value of $\hat{A}$

We can read only the value of $\hat{M}$

The final state $|\psi_{\mathrm{fin}}\rangle$ is an eigenstate $|b\rangle$ of $\hat{B}$.

We assume asymptotic behavior in $g \to 0$

$$\hat{U}_g \to \hat{1} - \frac{i}{\hbar} \hat{A} \otimes \hat{P}_M, \qquad \left[\hat{M}, \hat{P}_M\right] = i\hbar\hat{1}$$

Then, $\quad \frac{d}{dg} \hat{U}_g{}^{\dagger} \left(\hat{1} \otimes \hat{M}\right) \hat{U}_g \to \hat{A} \otimes \hat{1}$

Lee and Tsutsui's formula for weak value

Expectation value of the meter observable conditioned by the yield of the final measurement:

$$\mathbf{E}\left[\widehat{M}|\widehat{B}=b\right] := \frac{\langle\psi\otimes\lambda|\widehat{U}^{\dagger}\left(\widehat{\Pi}_b\otimes\widehat{M}\right)\widehat{U}|\psi\otimes\lambda\rangle}{\langle\psi\otimes\lambda|\widehat{U}^{\dagger}\left(\widehat{\Pi}_b\otimes\hat{1}\right)\widehat{U}|\psi\otimes\lambda\rangle}$$

Sensitivity of the meter

$$\lim_{g\to 0}\frac{d}{dg}\mathbf{E}\left[\widehat{M}|\widehat{B}=b\right] \quad = \quad \mathbf{Re}\frac{\langle\psi|\widehat{\Pi}_b\hat{A}|\psi\rangle}{\langle\psi|\widehat{\Pi}_b|\psi\rangle}$$

$$+\mathbf{Im}\left(\frac{\langle\psi|\widehat{\Pi}_b\hat{A}|\psi\rangle}{\langle\psi|\widehat{\Pi}_b|\psi\rangle}\right)\frac{1}{\hbar}\left\{\tfrac{1}{2}\langle\psi|\left(\widehat{M}\widehat{P}_M+\widehat{P}_M\widehat{M}\right)|\psi\rangle-\langle\psi|\widehat{M}|\psi\rangle\langle\psi|\widehat{P}_M|\psi\rangle\right\}$$

Lee and Tsutsui, PTEP (2017), Eq. (4.63)

Rewriting the weak value

If the eigenvalue of the final yield is non-degenerated, $\hat{\Pi}_b = |b\rangle\langle b|$ is a projection to 1-dim eigenspace. In this case, we have

$$\frac{\langle\psi|\hat{\Pi}_b\hat{A}|\psi\rangle}{\langle\psi|\hat{\Pi}_b|\psi\rangle} = \frac{\langle\psi|b\rangle\langle b|\hat{A}|\psi\rangle}{\langle\psi|b\rangle\langle b|\psi\rangle} = \frac{\langle b|\hat{A}|\psi\rangle}{\langle b|\psi\rangle} = \frac{\langle\psi_{\mathrm{fin}}|\hat{A}|\psi_{\mathrm{ini}}\rangle}{\langle\psi_{\mathrm{fin}}|\psi_{\mathrm{ini}}\rangle}$$

The formula of Lee and Tsutsui is reduced to the formula of Aharanov-Albert-Vaidman.

Rewriting the weak value 2

By putting $\hat{A}=\sum_a a\hat{\Pi}_a$ into,

$$\frac{\langle\psi|\hat{\Pi}_b\hat{A}|\psi\rangle}{\langle\psi|\hat{\Pi}_b|\psi\rangle}=\sum_a a\frac{\langle\psi|\hat{\Pi}_b\hat{\Pi}_a|\psi\rangle}{\langle\psi|\hat{\Pi}_b|\psi\rangle}$$

Probability formula of Born

$$\mathbf{P}\left[\hat{B}=b\right]:=\langle\psi|\hat{\Pi}_b|\psi\rangle$$

Pseudoprobability of Kirkwood-Dirac (complex number)

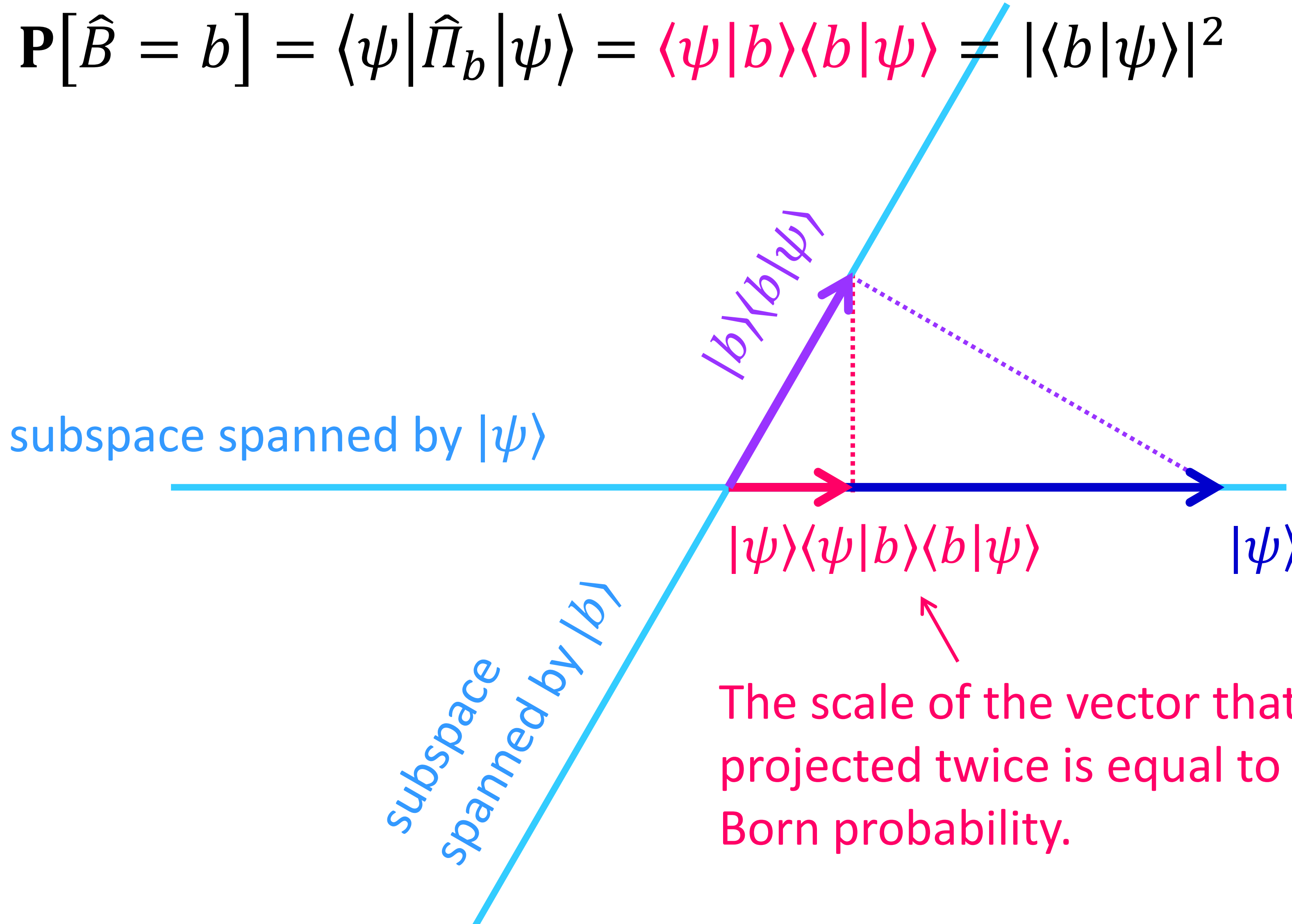
$$\mathbf{P}\left[\hat{A}=a,\hat{B}=b\right]:=\langle\psi|\hat{\Pi}_b\hat{\Pi}_a|\psi\rangle$$

Conditional probability

$$\mathbf{P}\left[\hat{A}=a\middle|\hat{B}=b\right]:=\frac{\mathbf{P}\left[\hat{A}=a,\hat{B}=b\right]}{\mathbf{P}\left[\hat{B}=b\right]}$$

Geometric interpretation of the Born probability

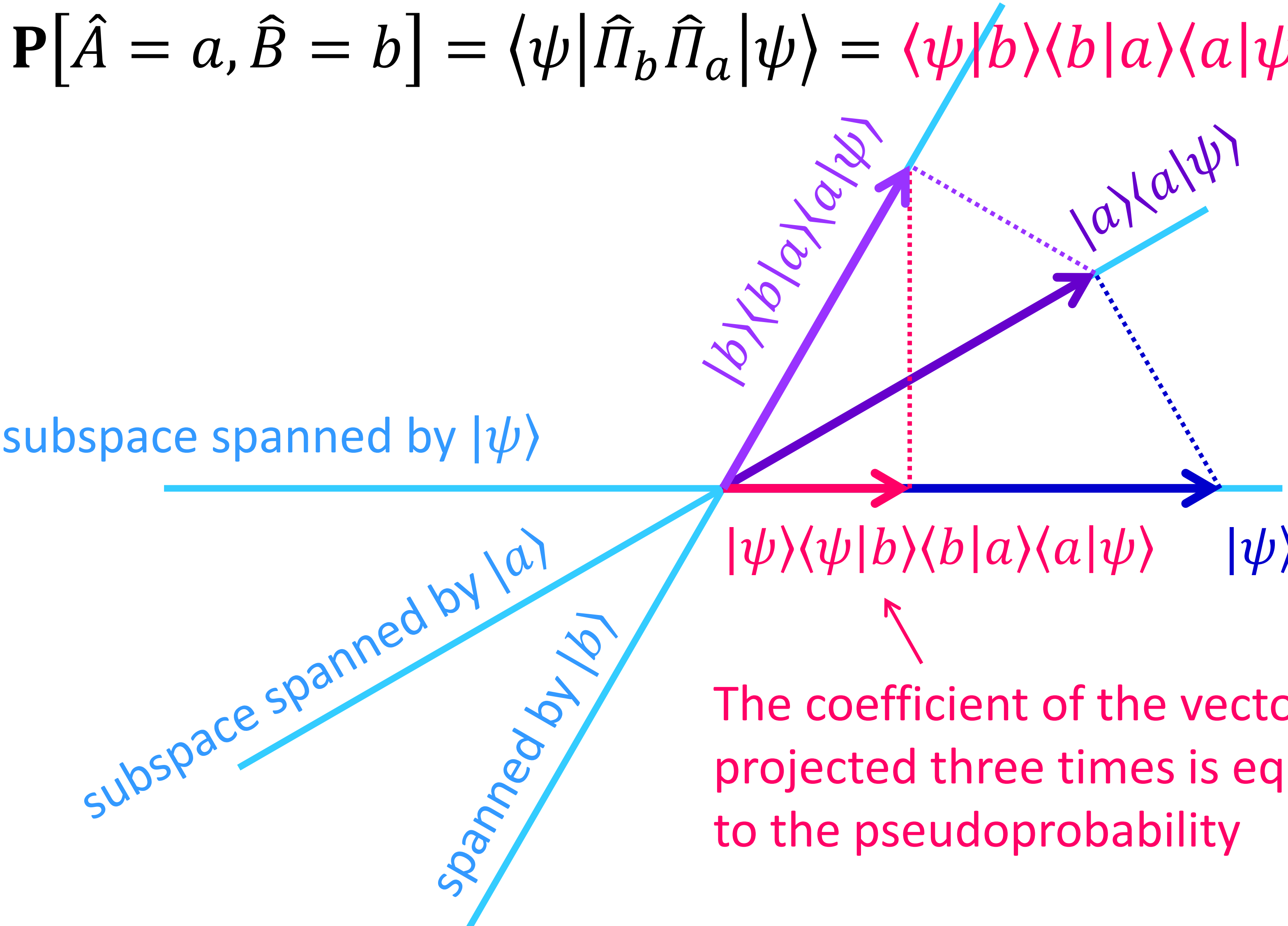
$$\mathbf{P}\left[\hat{B} = b\right] = \langle\psi|\hat{\Pi}_b|\psi\rangle = \langle\psi|b\rangle\langle b|\psi\rangle = |\langle b|\psi\rangle|^2$$



The scale of the vector that is projected twice is equal to the Born probability.

Geometric interpretation of the Kirkwood-Dirac pseudoprobability

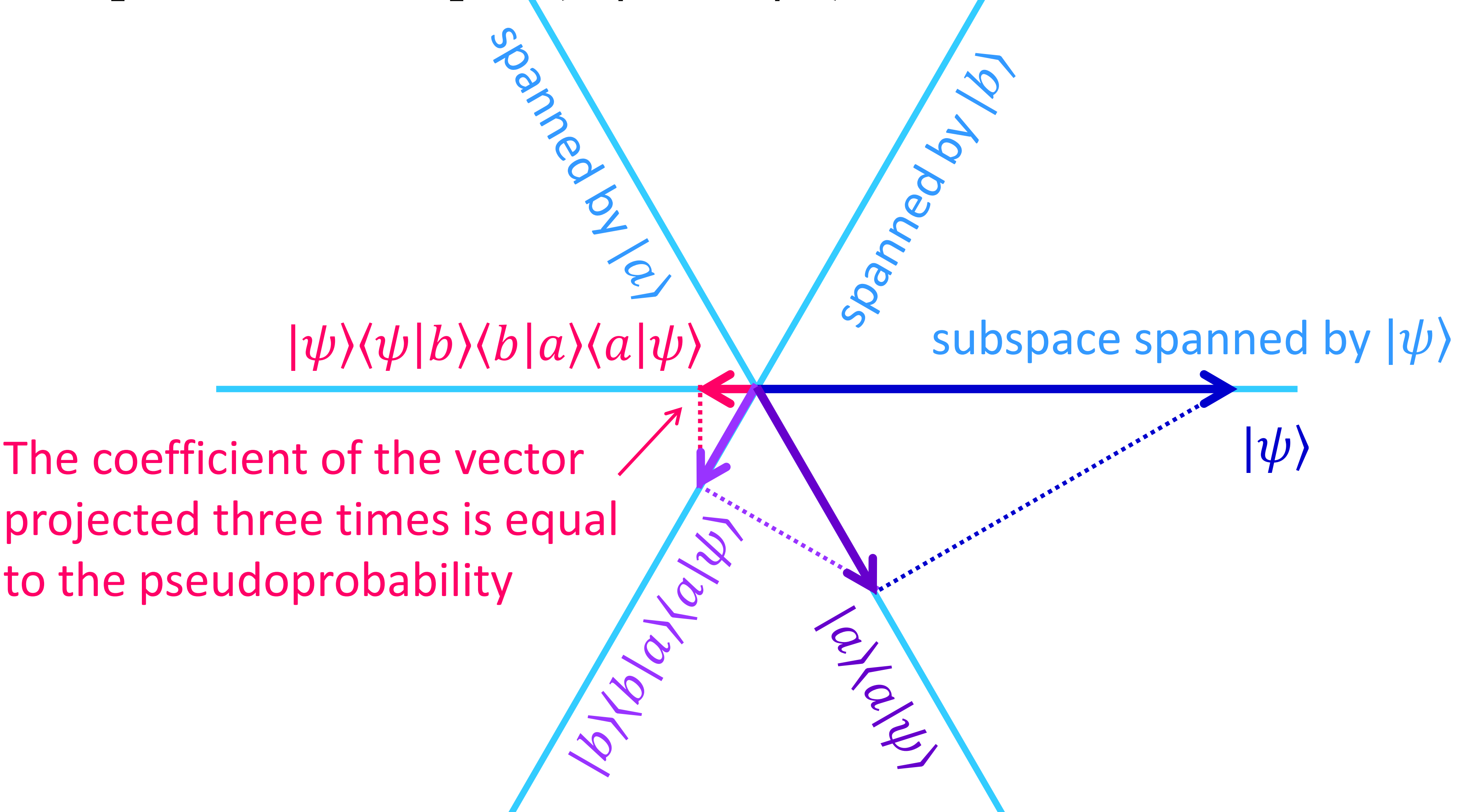
$$\mathbf{P}\left[\hat{A}=a,\hat{B}=b\right]=\langle\psi|\hat{\Pi}_b\hat{\Pi}_a|\psi\rangle=\langle\psi|b\rangle\langle b|a\rangle\langle a|\psi\rangle$$



The coefficient of the vector projected three times is equal to the pseudoprobability

When the pseudoprobability becomes negative

$$\mathbf{P}\left[\hat{A}=a,\hat{B}=b\right]=\langle\psi|\hat{\Pi}_b\hat{\Pi}_a|\psi\rangle=\langle\psi|b\rangle\langle b|a\rangle\langle a|\psi\rangle$$

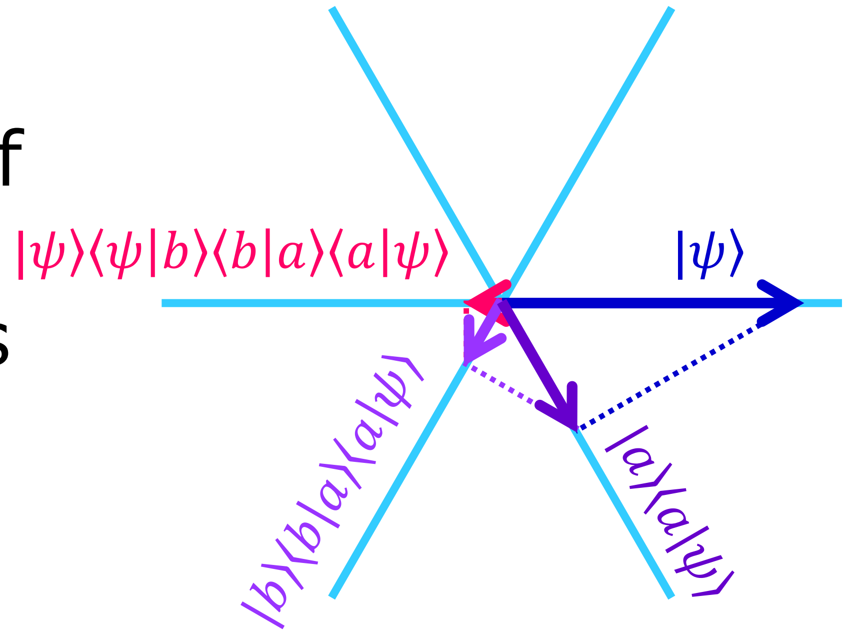


When the pseudoprobability becomes negative

- Commutative self-adjoint operators are simultaneously diagonalizable.
- Their eigenspaces are mutually parallel or orthogonal.
- The associated projections are commutative and their product is also projection.
- The joint probability $\mathbf{P}\left[\hat{A}=a, \hat{B}=b\right] = \langle\psi|\hat{\Pi}_b\hat{\Pi}_a|\psi\rangle = \langle\psi|\hat{\Pi}_a\hat{\Pi}_b|\psi\rangle$ is real and nonnegative.

When the pseudoprobability becomes negative?

- Noncommutative self-adjoint operators are not simultaneously diagonalizable.
- Their eigensubspaces are neither parallel nor orthogonal.
- It is possible to invert the direction of vectors by a sequential operations of non-orthogonal projections.

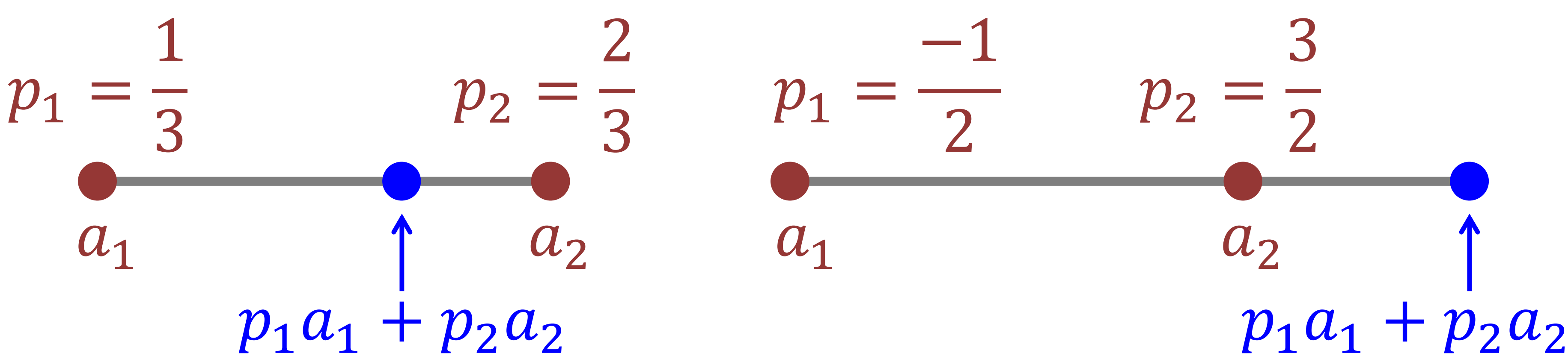


Necessary condition for obtaining negative pseudoprobability

- The object observable $\hat{A}$ is weakly measured at the intermediate state.
- The object observable $\hat{B}$ is projectively measured at the final state.
- It is necessary for obtaining negative pseudoprobability that the two observables $\hat{A}$ and $\hat{B}$ are noncommutative.

Negative probability induces amplification of weak value

- Expectation value = weighted average of spectral values
- If weights are positive, average is an internal dividing point.
- If some of weights are negative, average is an outer dividing point.



Understanding aided by pseudoprobability

- Violation of Bell’s inequality (Fine’s theorem)
- Complex pseudoprobability causes the Pancharatnam phase.
- Joint probabilities for initial, intermediate, and final states may define quantum stochastic process. But not yet fully discussed.

References

1. 谷村省吾「アインシュタインの夢ついえる」（ベルの不等式の破れの検証実験の解説記事）日経サイエンス2019年2月号のウェブ補足解説.
2. Lee and Tsutsui, “Quasi-probabilities in conditioned quantum measurement and a geometric/statistical interpretation of Aharonov’s weak value”, PTEP (2017).
3. Tamate, Kobayashi, Nakanishi, Sugiyama, Kitano, “Geometrical aspects of weak measurements and quantum erasers”, NJP (2009).

Thank you for your attention