Superselection Rules from Measurement Theory

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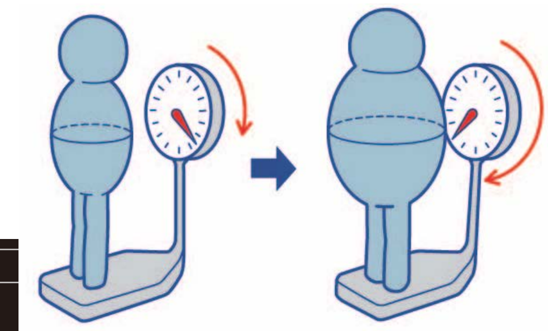
My recent popular articles

"Paradox of Photon" Nikkei Science, 2012 March

"New Uncertainty Relation," 2012 April

"What is Measurable?," 2012 July



Plan of this talk

- Introduction
- A toy model: momentum superselection rule
- Tool: von Neumann's indirect measurement model
- Basic notions: isolated conservation law, covariant indicator
- Main theorem: we derive the superselection rule from a conservation law in measurement process.

Superselection Rule

- J: superselection charge
- A : self-adjoint operator, $A^\dagger = A$

The superselection rule states

$$A \text{ is measurable} \;\Rightarrow\; [A, J] = 0$$

By contraposition,

$$[A, J] \neq 0 \;\Rightarrow\; A \text{ is non-measurable}$$

The superselection rule is a necessary condition for a self-adjoint operator A to be a measurable quantity.

History of superselection rule

Wick, Wigner, Wightman (1952) noticed that not every self-adjoint operator represents a physically measurable quantity.

ψ : Dirac field operator

$$\frac{1}{2}(\psi + \psi^\dagger), \quad \frac{1}{2i}(\psi - \psi^\dagger)$$

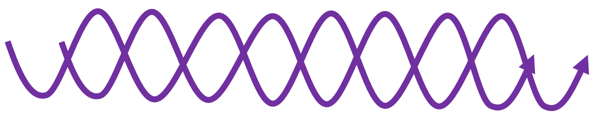
These are self-adjoint but they are not measurable.

$$\psi^\dagger \psi, \quad \bar{\psi}\gamma^\mu \psi$$

Charge density and current density are measurable.

Although the intensity of electron wave is measurable, its phase is non-measurable.

We can observe an interference fringe of the electron wave but we cannot determine its phase.



Univalence superselection rule

another derivation by Hegerfeldt, Kraus, Wigner (1968)

$J = R(2\pi)$: rotation by 360 degree around any axis.

A measurable quantity A must satisfy

$$R(2\pi)^\dagger A R(2\pi) = A \ \text{ or equivalently, } \ [A, J\,] = 0$$

On the other hand, the Dirac spinor field ψ satisfies

$$R(2\pi)^\dagger \psi R(2\pi) = -\psi, \qquad R(2\pi)^\dagger \psi^\dagger R(2\pi) = -\psi^\dagger$$

Thus the Dirac spinor field ψ itself is not a measurable quantity even though $\psi^\dagger \psi$ is measurable.

How did they notice it?

Around 1950, physicists discussed definition of the parity transformation of the Dirac field. It was not uniquely defined but it had an ambiguity.

Parity transform: $\psi(\boldsymbol{x},t) \rightarrow \Pi\psi(\boldsymbol{x},t) = e^{i\theta}\gamma^0\psi(-\boldsymbol{x},t)$

The phase factor $e^{i\theta}$ is not uniquely determined.

In 1952, WWW noted that the parity transformation of the Dirac spinor is allowed to be unfixed since the Dirac spinor itself is non-measurable.

Correspondence of mathematical notion to physical observable

- Mathematical notion: self-adjoint operator
- Physical notion:

  observable (measurable quantity)

Do they have one-to-one correspondence?

In the usual framework of quantum mechanics, their one-to-one correspondence is assumed.

von Neumann's argument (1932)

After showing that every observable is representable by a self-adjoint operator, von Neumann argued that it is appropriate to assume that there is a physical observable corresponding to each self-adjoint operator.

observable $\Rightarrow$ ($\Leftarrow$?) self-adjoint: $A^\dagger = A$

The superselection rule tells that this assumption is not appropriate.

There is a self-adjoint operator that does NOT correspond to any physical observable.

von Neumann, “Mathematical Foundations of Quantum Mechanics,” Section IV.2

- **井上・広重・恒藤 訳（1957年） p.250**

  《量子力学的系の物理量に対して超極大なエルミート作用素を一意的に対応させられることは，我々の知っている通りであるが，それに加えて，これらの対応は一対一である，すなわち，すべての超極大エルミート作用素は現実に物理量に対応している，と仮定するのが都合がよい．》
- Original German expression by von Neumann (1932)

  《... es ist zweckmässig anzunehmen, ...》
- English translation by Beyer (1955)

  《... it is convenient to assume, ...》
- English translation by Wightman (1995)

  《... it is appropriate to assume, ...》

Other examples:
Not every self-adjoint operator corresponds to observable

- Lorentz boost generators
- Dilatation generator

For making a clear argument we need a clear definition

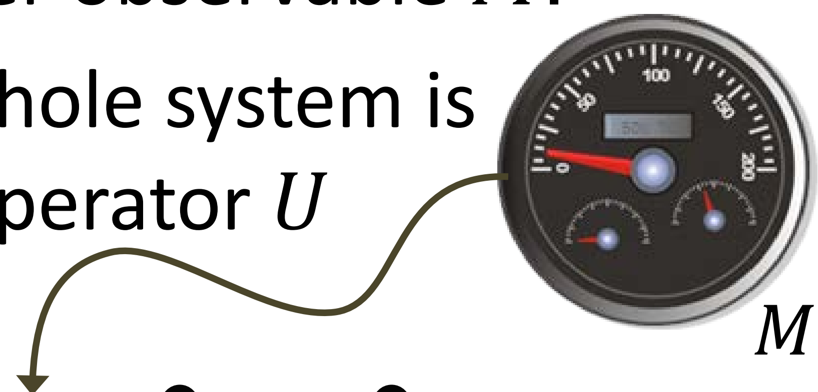
- self-adjoint: mathematically well-defined notion
- observable (something measurable): not clear

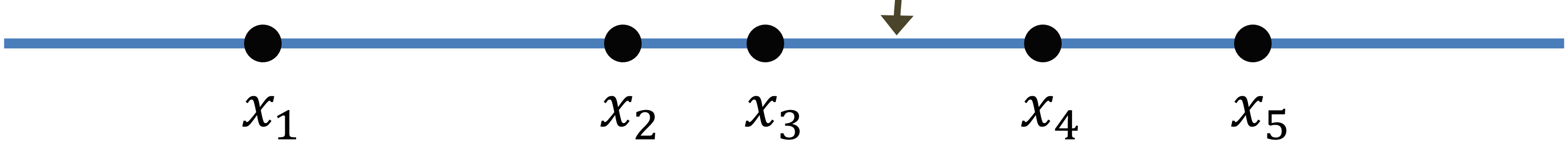
⇩

It is necessary to formulate the notion of measurement.

Toy model to help understanding of the superselection rule

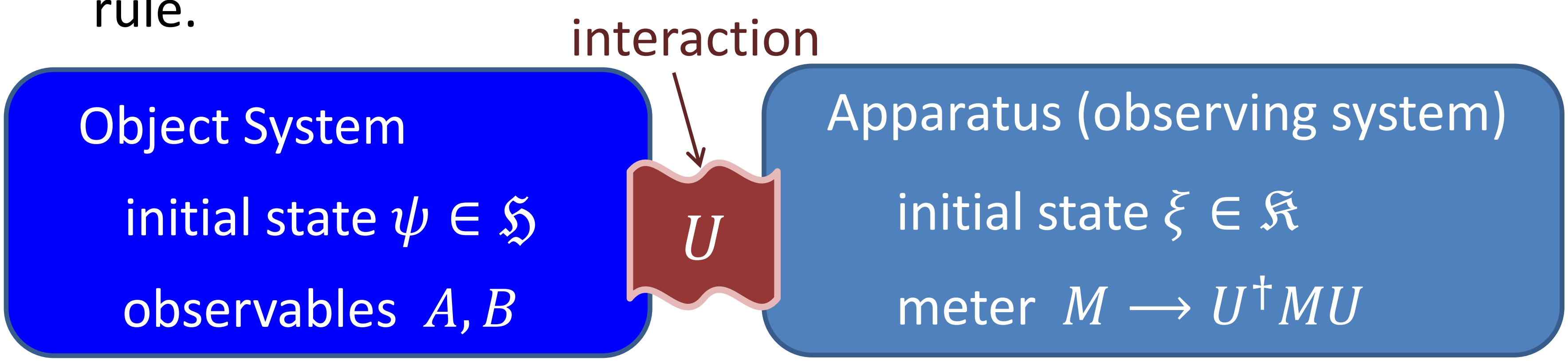
- n particles in one-dimensional space
- masses: $m_1, m_2, \dots, m_n$
- positions: $x_1, x_2, \dots, x_n$
- momenta: $p_1, p_2, \dots, p_n$
- An apparatus has a meter observable M.
- Time-evolution of the whole system is described by a unitary operator U





Indirect measurement model

- The object system (n-particle system) has a Hilbert space $\mathfrak{H}$, while the apparatus has a Hilbert space $\mathfrak{K}$.
- The initial state of the whole system is $\psi \otimes \xi \in \mathfrak{H} \otimes \mathfrak{K}$
- An observable A to be measured is a self-adjoint operator on $\mathfrak{H}$, while a meter observable M is a self-adjoint operator on $\mathfrak{K}$.
- Interaction between them is described by a unitary operator $U = e^{-iHt/\hbar}$ on the composite system $\mathfrak{H} \otimes \mathfrak{K}$.
- The meter $U^{\dagger} M U$ is read out by means of the Born probability rule.



Requirement

- Suppose that we want to measure the position of the center of mass of the particles:

$$A = X := \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i}$$

- Assume that the total momentum of the particles is conserved during the measurement process (isolated conservation law):

$$J = P := \sum_{i=1}^{n} p_i \,, \qquad U^{\dagger} P U = P$$

- Does the meter move as $U^{\dagger} M U = M + X$?
- Answer: It is impossible.

Proof

The total momentum of the particles $P = P \otimes 1$ and the meter position operator $M = 1 \otimes M$ on $\mathfrak{H} \otimes \mathfrak{K}$ commute

$$[P, M] = 0$$

Since the time-evolution acts as an automorphim,

$$[U^\dagger P U, U^\dagger M U] = U^\dagger [P, M] U = 0$$

On the other hand, the momentum conservation and the meter shift condition imply

$$[U^\dagger P U, U^\dagger M U] = [P, M + X] = [P, X] = -i\hbar$$

These give a contradiction. In general, a quantity A measurable in the sense $U^\dagger M U = M + A$ must satisfy $[P, A] = 0$.

General scheme which gives a rise of superselection rule

definition of measurability

interaction

Object System

observable A

isolated conserved quantity $J \longrightarrow U^{\dagger} J U = J$

U

Apparatus (observing system)

covariant meter

$$M \longrightarrow U^{\dagger} M U = M + A$$

Derivation of the superselection rule

$$
\begin{aligned}
&[J, M] = 0, \\
&0 = U^{\dagger}[J, M]U \\
&\quad = \left[U^{\dagger} J U, U^{\dagger} M U\right] \\
&\quad = [J, M + A] \\
&\quad = [J, A] \\
&\therefore [J, A] = 0.
\end{aligned}
$$

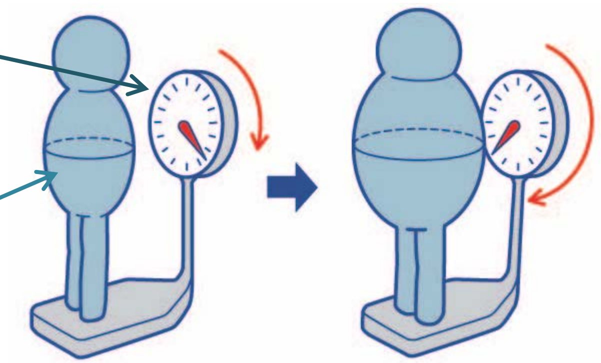
A more general proof is given in my paper.

Measurability

- Physically meaningful measurement requires covariance between the quantity to be measured and the quantity to be read out.

meter M
: quantity to be read

human weight A
: quantity to be known



meter $M \rightarrow M'$

weight $A \rightarrow A'$

covariance is necessary for making meaningful measurement.

- But the structure of interaction between the object system and the apparatus may or may not allow the covariance.

Measurable/non-measurable quantities

The superselection rule associated to the momentum conservation law demands any measurable quantity A must satisfy $[P, A] = 0$, where $P = \sum_i p_i$ is the total momentum.

Measurable: relative coordinates

$$A = x_r - x_s, \qquad A = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} - x_3$$

Non-measurable: absolute coordinates of the particle

$$A = x_r, \qquad A = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i}$$

Symmetries / Superselection charges

- 2π rotation invariance / univalence

  The Dirac spinor, which is not invariant under 2π rotation, is non-measurable.

- U(1) invariance / electric charge, baryon number

  The phases of matter waves of electron or neutron are non-measurable.

- Notice: Photon number is not conserved, hence, the phase of electromagnetic wave is measurable.

Paradox associated with Non-abelian symmetry

For example, SO(3) rotation invariance implies the conservation of angular momenta J_x, J_y, J_z, which are noncommutative each other.

Hence the superselection rule prohibits the measurements of angular momenta.

But, in actual experiments, we measure the spin angular momenta of electrons or photons.

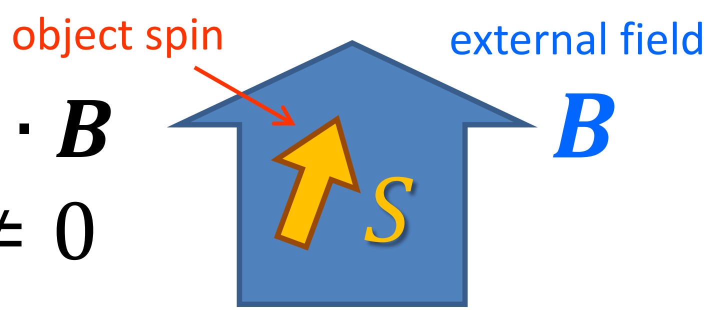
How is it possible?

Solution of the angular momentum paradox

The SO(3) rotation invariance is broken by introducing external magnetic field (Zeeman effect or Stern-Gerlach setting) for nuclei or electrons, polarization filter or birefringent crystal for photons.

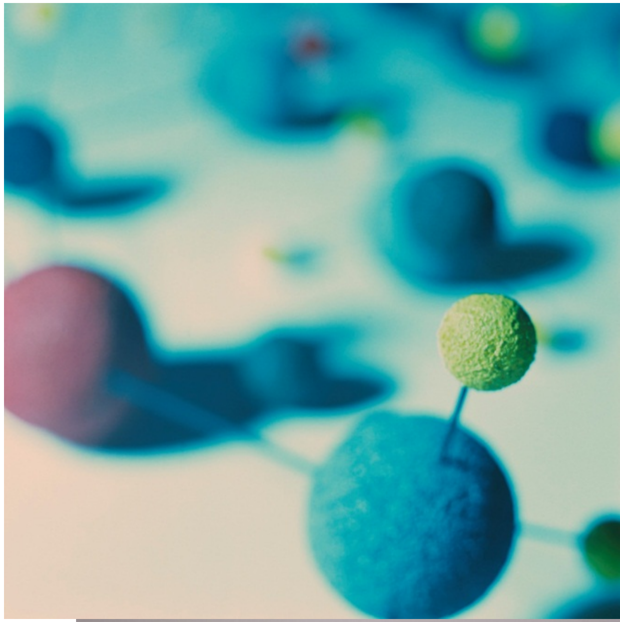
All of the measurements of angular momenta introduce a coupling of the object system and the apparatus that breaks the isolation of the system and allows exchange of angular momenta between the two systems.

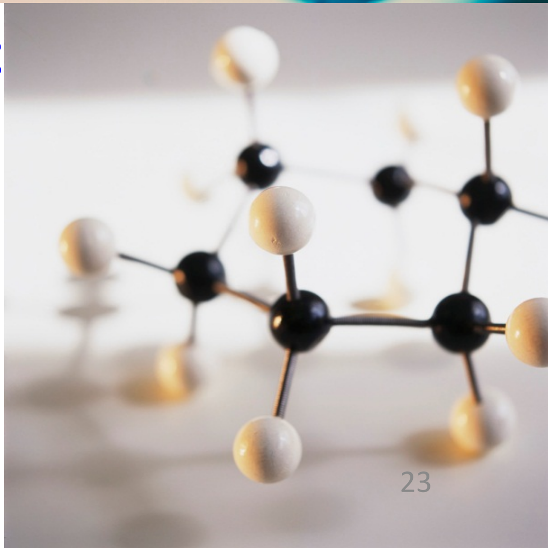
$$H = g\boldsymbol{S}\cdot\boldsymbol{B}$$
$$[H, \boldsymbol{S}] \neq 0$$



$$H = g\boldsymbol{S}\cdot\boldsymbol{L}$$
$$[H, \boldsymbol{S}] \neq 0$$
$$[H, \boldsymbol{S}+\boldsymbol{L}] = 0$$

Why we can measure rotationally non-invariant quantities?





If the SO(3) rotation invariance is preserved within a microscopic system, we cannot measure any rotationally variant quantity from outside.

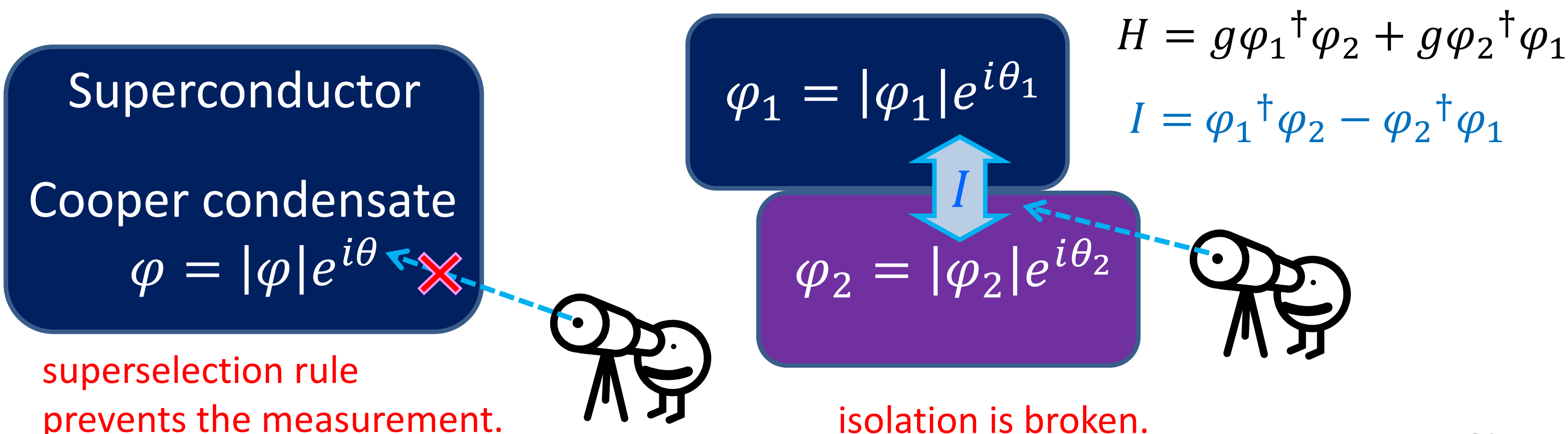
However, actually we can measure it since the rotation invariance is spontaneously broken at the macroscopic scale. We can construct apparatus which has a non-spherical shape. Thus, we can apply rotationally non-invariant external field on a microscopic system.

How can we overcome the U(1) superselection rule?

The isolated conservation of the U(1) charge makes measurements of gauge variant quantities impossible.

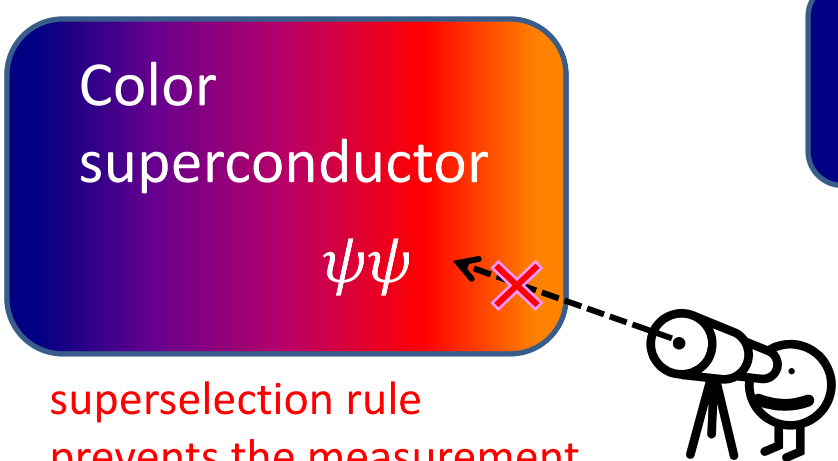
By breaking the isolation, we can make such a measurement possible.

Example: superconductivity, Josephson junction

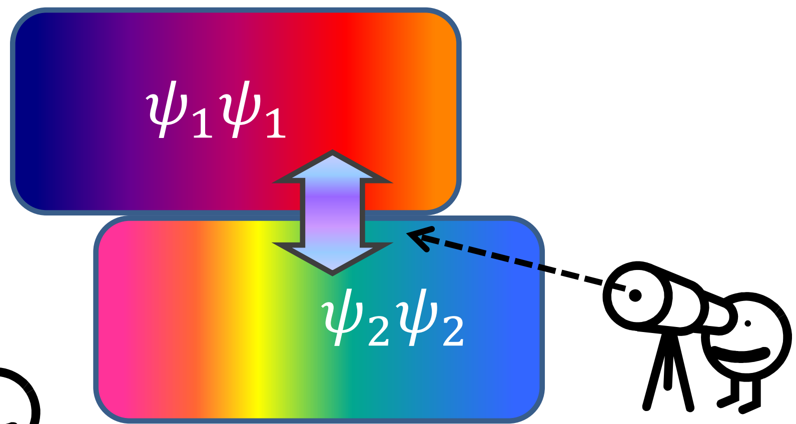


Why the SU(3) color is invisible?

In QCD, colored quantities like quark and gluon fields are non-measurable from outside of hadrons. Moreover, since SU(3) is non-abelian, color charge itself is non-measurable. If there are objects in which the color symmetry are spontaneously broken, we can measure their relative color difference.



superselection rule
prevents the measurement.



isolation is broken.

Uncertainty relation under a conservation law: Wigner-Araki-Yanase-Ozawa theorem

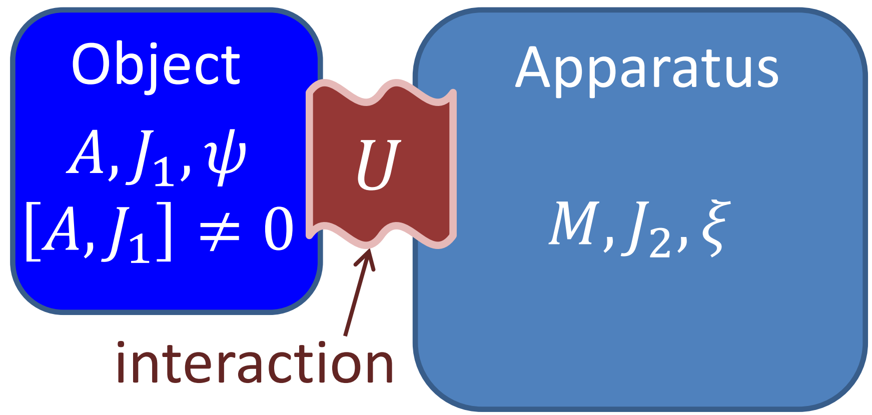
A : observable to be measured

M : meter of the apparatus

J_1 : quantity of the object system

J_2 : quantity of the apparatus

U : unitary time-evolution



$\Omega = \psi \otimes \xi$: initial state of the whole system

$\varepsilon(A)^2 = \left\langle \Omega \left| \left(U^\dagger M U - A \right)^2 \right| \Omega \right\rangle$: measurement error

$\sigma(J_1)^2 = \langle \Omega | (J_1)^2 | \Omega \rangle - \langle \Omega | J_1 | \Omega \rangle^2$: standard deviation

Assume $U^\dagger (J_1 + J_2) U = J_1 + J_2$

Theorem: $$\varepsilon(A)^2 \geq \frac{|\langle [A, J_1] \rangle|^2}{4\{\sigma(J_1)^2 + \sigma(J_2)^2\}^2}$$

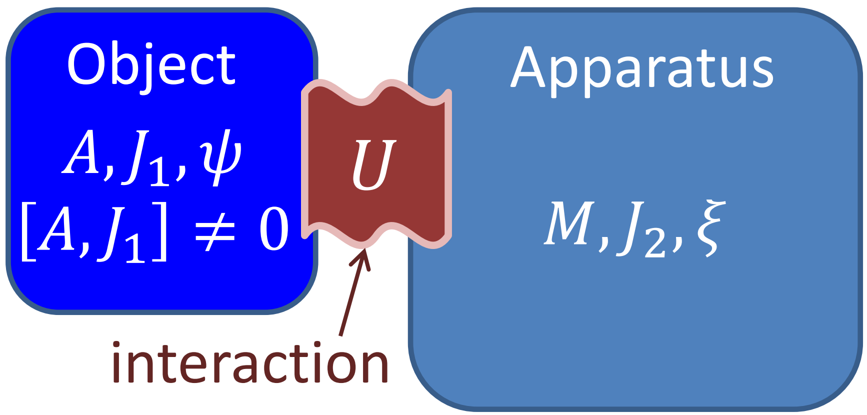
Comparison of the WAY-Ozawa theorem with the superselection rule

The WAY theorem:

The conservation of total charge

$$U^\dagger(J_1 + J_2)U = J_1 + J_2$$

implies that a non-vanishing error of the measurement $\varepsilon(A)$ is evitable.



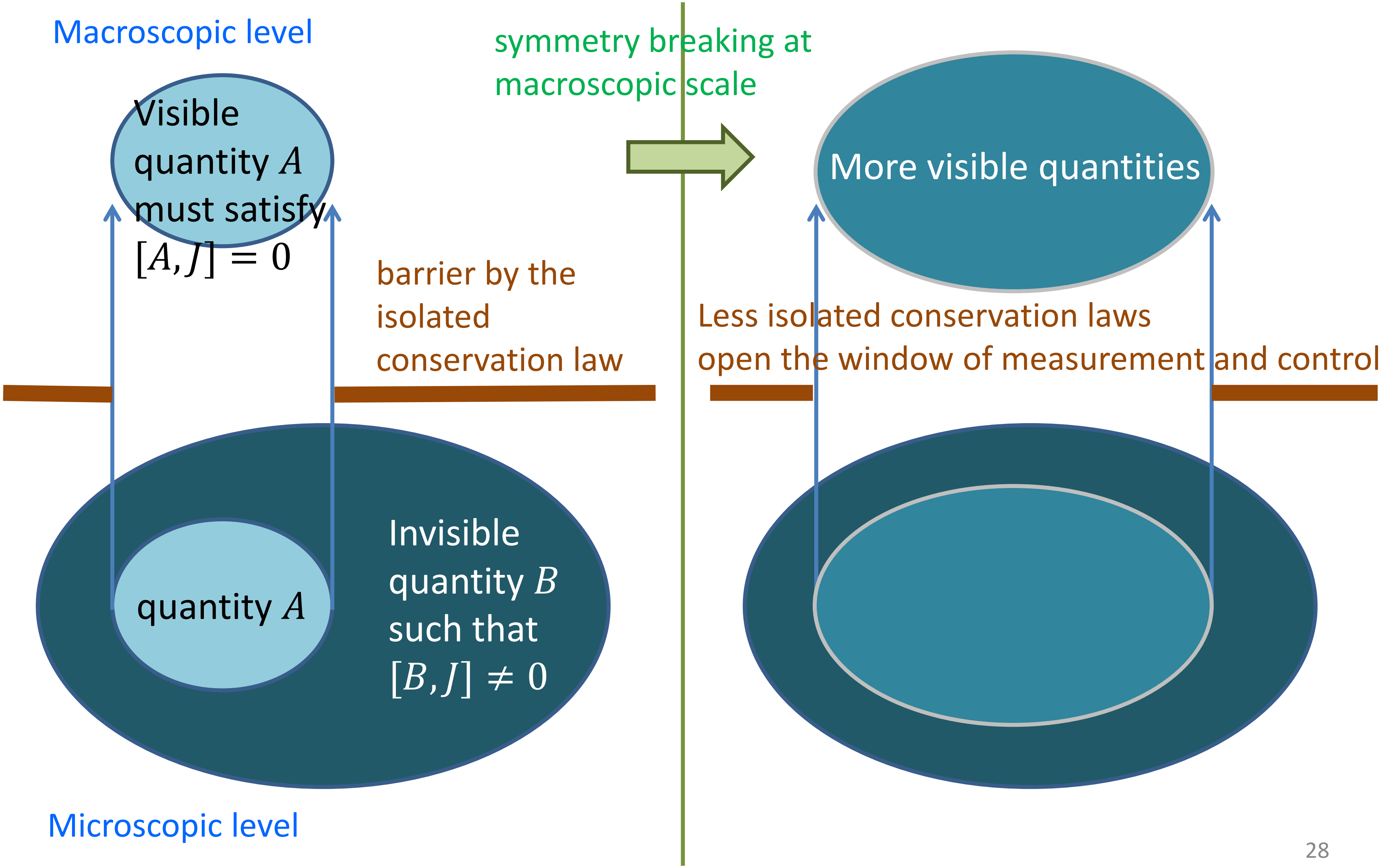
Superselection rule: The isolated conservation law

$$U^\dagger J_1 U = J_1$$

implies the impossibility of $M \longrightarrow U^\dagger M U = M + A$, that is, the meter M cannot move covariantly to the object quantity A.

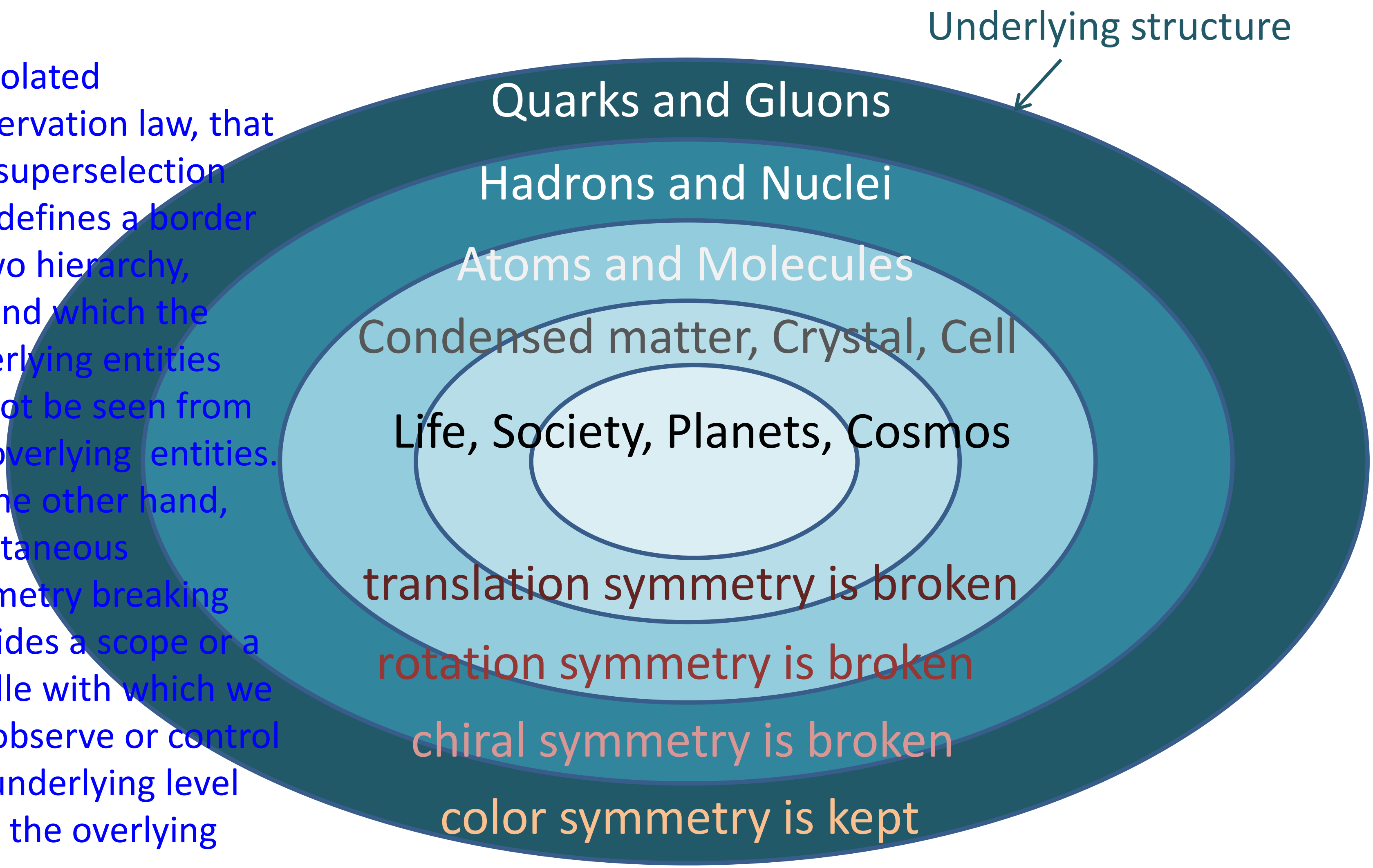
This is an extremal form of the error-disturbance uncertainty relation.

Accessible and Inaccessible Levels of the World



Hierarchical structure of the nature

An isolated conservation law, that is, a superselection rule defines a border of two hierarchy, beyond which the underlying entities cannot be seen from the overlying entities. On the other hand, spontaneous symmetry breaking provides a scope or a handle with which we can observe or control the underlying level from the overlying level.



a picture inspired by P. W. Anderson's "More is different"

Conclusion

- The uncertainty relation tells that we cannot precisely measure a quantity A without disturbing another quantity J such that $[A, J] \neq 0$.
- If the disturbance of J is absolutely prohibited, namely, if the object system has the isolated conserved quantity J, the measurement of A becomes impossible. This is the superselection rule.
- The superselection rule is understood as a consequence of symmetry from a viewpoint of measurement theory.
- We can overcome the superselection rule by introducing explicit or spontaneous symmetry breakings.
- Isolated conservation laws and spontaneous symmetry breakings build the hierarchical structure of the nature.

The End