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Clustering behavior of time-delayed nearest-neighbor coupled oscillators

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We analyze both numerically and theoretically the dynamical behaviors of systems composed of limit-cycle oscillators, which are coupled by a time-delayed nearest-neighbor interaction. In this system we found cluster states which are characterized by a phase difference of neighboring oscillators and their collective frequency. A time-delay turns out to have an important effect on the stability of various cluster states.

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I. INTRODUCTION

Quite a few papers have been devoted to the dynamics of populations of coupled oscillators. Such populations are known to model many systems in physics [1,2], biology [3–6], and chemistry [7]. It has been shown that coupled limit-cycle oscillators with weak coupling can be described by a system of phase-coupled oscillators, where each oscillator is described by a single variable, its phase [7]. Therefore, if we could show some features for a system of phase coupled oscillators, we would be able to expect that the same is true for more general oscillator systems.

One of the remarkable phenomenon exhibited by a system of coupled oscillators is synchronization or entrainment; when oscillators, not necessarily identical to each other, are allowed to mutually interact, it sometimes happens that the dynamics of oscillators are entrained to an attractor in which every oscillator performs a synchronized (frequency- or phase-locked) oscillation. It is reported that when the natural frequency of each oscillator is distributed around an average one, there exists a critical strength of coupling, above which oscillators can be synchronized [8–10].

Furthermore an attractor, which represents a so-called cluster state, has been found for an oscillator system with mean-field coupling, i.e., a globally coupled oscillator system [11,12]. This state is characterized by the coexistence of subgroups each of which consists of fully phase-synchronized oscillators. This clustering phenomenon is interesting from the standpoint of a symmetry breaking because a collection of dynamical elements is spontaneously divided into some groups. In a sense the entrained or phase-locked state, mentioned before, may be called a one-cluster state. The opposite extremity is the antiphase state [13,14] in which all the elements of the system (the number is N) oscillate with the same frequency and with equal spacing $2\pi/N$ in phase (a N-cluster state). Thus a system can have various cluster states.

In this paper we mainly investigate clustering behaviors in a phase-coupled oscillator system. The model systems may be variously classified even if we confine our discussion to phase-coupled systems. First, if we pay attention to the range of interaction among oscillators, the models may belong to either the mean-field (MF) [8,11,12] or short-range (SR) [9,10,15–18,20] interaction model. Due to the simplification of the analysis in the case of MF coupling, the MF model has attracted much attention. However, spatial structures, which are absent in the MF model, are expected to yield a variety of self-organized phenomena in a system with SR coupling. From this point of view we are mainly concerned with the SR model. Second, we now turn to the time delay of the interaction [19–21]. In models which are related to physical and/or chemical phenomena, a time delay in the interaction among oscillators does not play an important role. However, in models which describe interaction among neurons or some biological elements, it is sometimes important to take into account effects of finite time τ required for information transmission between two elements. Therefore we consider both cases $\tau=0$ and $\tau\neq0$.

Summarizing the above we study the collective behavior of the time-delayed nearest-neighbor (NN) coupled oscillator systems. Such systems have been studied by some authors. It is reported that due to the time delay there exists a multitude of synchronized solutions [19] and that time delay leads to a frequency depression [20]. However, it has not been reported whether or not clustering behavior can be observed in the NN coupling systems and how the stability of cluster states is modified under the influence of a finite time delay if such states exist. This is the main topic of this paper.

This paper is organized as follows. In Sec. II we present our model and in Sec. III results of our computer experiments are reported with the main emphasis put on the clustering behavior for the NN coupling systems. In Sec. IV some theoretical studies are performed with use of both an energy principle for the case $\tau=0$ and the linear stability analysis for the case with a finite time delay ($\tau\neq0$). Section V contains some remarks on generalization to higher dimensional systems and summary of the results obtained in this paper.

II. MODEL

In this section we consider a one-dimensional system composed of N phase-coupled oscillators whose dynamics are described as

$$\frac{d\phi_k(t)}{dt}=\omega_0+\frac{K}{I}\sum_j \sin\{\phi_j(t-\tau)-\phi_k(t)\} \quad (k=1,\ldots,N)\ , \tag{1}$$

where $\phi_k(t)$ denotes the phase of the kth oscillator at time t and ω_0 the common natural frequency, i.e., the frequency in the absence of interaction among oscillators. K is the coupling strength, I is the number of oscillators with which one oscillator interacts, and τ is a time delay. In Sec. V we will comment on the extension to a higher dimensional model. Since we are mainly concerned with the NN coupling, the summation variable j takes the value $k-1$ and $k+1$, and $I=2$. We also consider, for the sake of comparison, the case of MF coupling for which j takes all the values except for k and $I=N-1$. We note that ϕ_{N+1} is identical to ϕ_1 due to the periodic boundary condition. Furthermore, by variable transformations

$$\omega_0 t=\tilde{t},\quad \frac{K}{\omega_0}=\tilde{K},\quad \omega_0\tau=\tilde{\tau},\quad \phi_k(t)=\tilde{\phi}_k(\tilde{t}) \tag{2}$$

we get the dimensionless equation

$$\frac{d\tilde{\phi}_k(\tilde{t})}{d\tilde{t}}=1+\frac{\tilde{K}}{I}\sum_j \sin\{\tilde{\phi}_j(\tilde{t}-\tilde{\tau})-\tilde{\phi}_k(\tilde{t})\} \quad (k=1,\ldots,N)\ . \tag{3}$$

In what follows, we consider Eq. (3) with tildes omitted for simplicity.

Systems with the MF coupling and no time delay ($\tau=0$) have been studied intensively. For $\tau=0$ it is natural to term the coupling with $K>0$ ($K<0$) as attractive (repulsive), because a small deviation $\Theta_{jk}=\phi_j-\phi_k$ tends to decrease (increase) under the interaction $K\sin\Theta_{jk}$. We use this terminology even in the case of finite τ. These points will be discussed in detail in Sec. IV when we study the stability of cluster solutions based on an energy analysis.

In anticipation of the various cluster state which are found in our computer simulation, and also for convenience of the later discussion, we give here a systematical rule to produce cluster states. Before proceeding to the case of the NN coupling we first explain the (symmetric) cluster states for the case of the MF coupling which are discussed in detail by Okuda [12]. A symmetric N_c-cluster state, where N_c is one of the divisor of N, is defined as the state in which each cluster l ($l=0,1,\ldots,Nc-1$), consisting of N/N_c oscillators with the same phase, rotates with the same frequency Ω. Thus, when the phase of N_c clusters are equally separated, the phase ψ_l of cluster l is described as

$$\psi_l=\Omega t+\frac{2\pi l}{N_c}\quad (l=0,\ldots,N_c-1)\ . \tag{4}$$

In the case of $\tau=0$, Ω is determined to be

$$\Omega=1+\frac{K}{N_c}\sum_{l=0}^{n-1}\sin\left[\frac{2\pi l}{N_c}\right]. \tag{5}$$

Since there are many ways of distributing N oscillators into N_c groups with equal number of elements ($=N/N_c$), the solution (4) and (5) is highly degenerate. The case of $N_c=N$ has attracted special attention because of the $(N-1)!$ attractors (a so-called attractor crowding) [13,14].

We now turn to the cluster state in the NN case. The model (3) is explicitly written as

$$\frac{d\phi_k(t)}{dt}=1+\frac{K}{2}\sum_{j=k\pm1}\sin\{\phi_j(t-\tau)-\phi_k(t)\}\ . \tag{6}$$

In this case the phase difference

$$\Phi=\phi_{k+1}(t)-\phi_k(t)\ , \tag{7}$$

which turns out to be independent of k and t, plays the fundamental role as N_c does in the case of MF model. Equation (7) means that, in constructing a cluster state characterized by Φ, we first specify the phase ϕ_1 of the first oscillator. Then we take $\phi_2=\phi_1+\Phi$ and so on. Since Φ denotes a phase, it is in the range $0\le\Phi<2\pi$. However, if Φ happens to be larger than π, $\Phi=\phi_k-\phi_{k+1}$ can be taken in the range $0\le\Phi\le\pi$ and we follow the prescription $\phi_N=\phi_1+\Phi$, $\phi_{N-1}=\phi_N+\Phi$, and so on. Thus hereafter we will restrict Φ in the range $0\le\Phi\le\pi$. With use of phase unit

$$\theta=\frac{2\pi}{N}\ , \tag{8}$$

Φ is expressed, from the condition $N\Phi=2\pi n$, as

$$\Phi=n\theta,\quad n=\begin{cases}0,\ldots,N/2 & \text{for } N \text{ even}\\ 0,\ldots,(N-1)/2 & \text{for } N \text{ odd}\ .\end{cases} \tag{9}$$

Thus, the number of the cluster N_c is determined, when $\Phi\neq0$, from the condition

$$\Phi N_c=n\theta N_c=2\pi m\ , \tag{10}$$

where m denotes the smallest integer that satisfies Eq. (10). For example, in the case of $N=30$, $N_c=10$ for $n=3$, $N_c=5$ for $n=12$, and so on. For $\Phi=0$, it is evident that $N_c=1$.

As to the dynamics of the cluster states, the phase of kth oscillator evolves in time as

$$\phi_k(t)=\Omega t+k\Phi\quad (k=1,\ldots,N)\ , \tag{11}$$

where Ω is the common frequency of the oscillators. Inserting Eq. (11) into Eq. (6), we get

$$\Omega=1-K\sin(\Omega\tau)\cos\Phi\ . \tag{12}$$

Even if Φ (or n) is given, depending on the parameter K and τ, the transcendental equation (12) for Ω may have more than one solution, which we denote as Ω_b ($b=1,\ldots,B$). Thus in order to unambiguously define a cluster state we must specify not only Φ, Eq. (7), but also Ω_b, and we denote the cluster state by (Φ, Ω_b).

III. COMPUTER EXPERIMENTS FOR NN COUPLING

We performed three types of computer experiments to examine (i) the structure of cluster states, (ii) the linear stability of the cluster states, and (iii) the relative nonlinear stability among the linearly stable cluster states. Computer experiments were performed by integrating Eq. (6) with the use of the Runge-Kutta-Gill algorithm with the time step $\Delta t = 0.01$. We have checked that decreasing the time step does not affect the results. We also examined systems consisting of different number of oscillators. Since general features of clustering behavior does not depend on N, we show results for $N = 30$ in the following. The initial conditions of the oscillators were chosen randomly or some cluster states according to the purpose of experiments.

A. Examples of cluster states

In this subsection we show some examples of clustering behavior of the system with $N = 30$, thus phase unit $\theta = \pi/15$ [see Eq. (8)]. For the initial condition we set

$$\phi_k(t) = 1(t+\tau) + \alpha_k \quad (-\tau \le t \le 0) , \tag{13}$$

where phases α_k $(k = 1, \ldots, N)$ are randomly distributed over $[0, 2\pi]$. For the attractive coupling $(K > 0)$ either a 1-cluster state $(\Phi = 0)$ or a 30-cluster state $(\Phi = \theta)$ is produced depending on the initial condition and for the repulsive coupling $(K < 0)$ cluster states with large Φ, either a 15-cluster state $(\Phi = 14\theta)$ or a 2-cluster state $(\Phi = 15\theta)$ is produced. Some examples of clustering behaviors are given in Fig. 1, which depicts trajectories of time evolution of oscillators $\phi_k(t)$ $(k = 1, \ldots, N)$. These cluster states are also produced in the case of finite time delay, as shown in Fig. 2, $(K, \tau) = (1, 1)$. As mentioned in Sec. III B, other linear stable cluster solutions can exist in the case studied above. However, we could not get such cluster states in our simulations. This suggests that the 1- or 30-cluster states in the case of attractive coupling and 2- or 15-cluster states in the case of repulsive coupling have the largest basin of attraction. When there is no time delay these results are physically explainable from an energy principle (Sec. IV). We also examine the time evolution of the average frequency $\langle \dot{\phi} \rangle = (1/N)\sum_k \dot{\phi}_k(t)$ for various time delays τ when a 1-cluster state is produced under the initial condition (13). It can be seen from Fig. 3 that the averaged frequency of clusters become small as the time delay τ becomes large. In the case of parameters studied in Fig. 3, Eq. (12) has only one solution, which is indicated by symbols in Fig. 3.

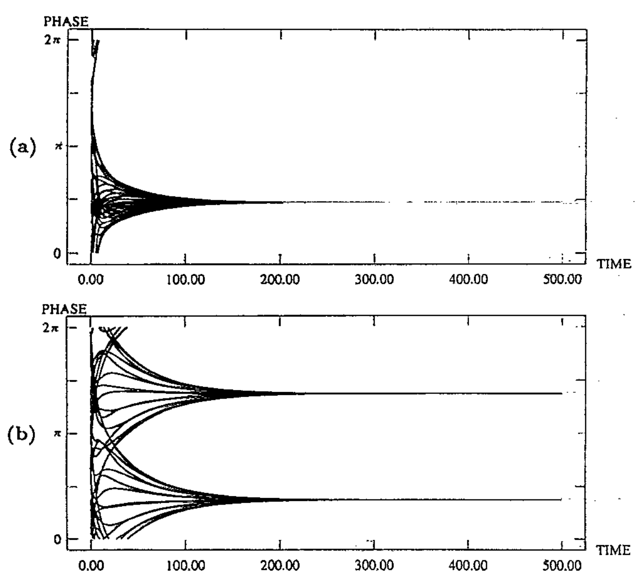

FIG. 1. Time evolution of oscillators $\phi_k(t) - \Omega t$ $(k = 1, \ldots, 30)$ in the case of no time delay. (a) 1-cluster state $(\Phi = 0)$ for $(K, \tau) = (1, 0)$ with $\Omega = 1.0$ from Eq. (12). (b) 2-cluster state $(\Phi = \pi)$ for $(k, \tau) = (-1, 0)$ with $\Omega = 1.0$ from Eq. (12).

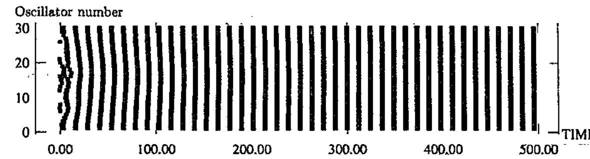

FIG. 2. Time evolution of oscillators in the case of a finite time delay for $(K, \tau) = (1, 1)$ with $\Omega = 0.510973$ from Eq. (12). The 1-cluster state $(\Phi = 0)$ is produced. The vertical axis denotes the number of oscillators. The dark region indicates that the phase takes a value between 0 and $\pi/4$.

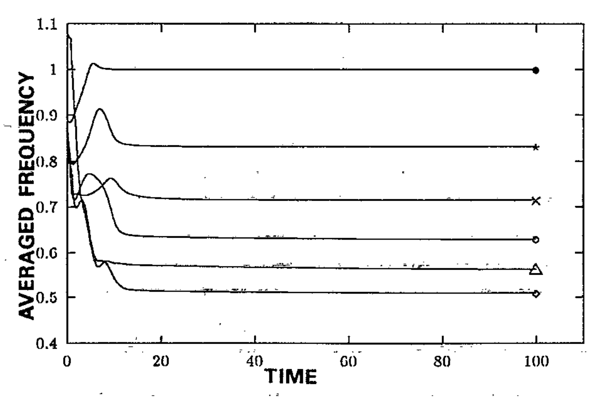

FIG. 3. Time evolution of averaged frequencies $(1/N)\sum_k \dot{\phi}_k(t)$ for various time delays (from the top $\tau = 0.0, 0.2, 0.4, 0.6, 0.8, 1.0$) $(K = 1)$. Increasing a time delay induces frequency depression. The symbols indicate the solutions of Eq. (12) [●, 1.0 $(\tau = 0)$; ★, 0.833977 $(\tau = 0.2)$; ×, 0.717084 $(\tau = 0.4)$; ○, 0.630602 $(\tau = 0.6)$; △, 0.563973 $(\tau = 0.8)$; ◊, 0.510973 $(\tau = 1.0)$], which precisely agree with the experiments.

TABLE I. Summary of computer experiments for linear stability with $K=1$. Here the phase unit $\theta=\pi/15$ and N_c denotes the number of cluster. S means the cluster is stable and U unstable.

Φ	N_c \ τ	0	1	2
0	1	S	S	S
θ	30	S	S	S
2θ	15	S	S	S
3θ	10	S	S	S
4θ	15	S	S	U
5θ	6	S	S	U
6θ	5	S	U	U
7θ	30	S	U	U
8θ	15	U	U	U
9θ	10	U	U	U
10θ	3	U	U	U
11θ	30	U	U	S
12θ	5	U	U	S
13θ	30	U	S	S
14θ	15	U	S	S
15θ	2	U	S	S

B. Linear stability of the cluster state

Next we study linear stability of various cluster states. For this purpose the initial condition is set as

$$\phi_k(t)=\Omega(t+\tau)+k\Phi+\varepsilon_k \quad (-\tau\le t\le 0)\ , \tag{14}$$

where ε_k $(k=1,\ldots,N)$ are random numbers in the range $[-\pi/100,\pi/100]$. This initial state is the one that is slightly perturbed from a cluster state (Φ). If this cluster state is restored in spite of a perturbation, then it is stable; otherwise this cluster state is unstable. Our simulation results are summarized in Table I. In this table S denotes that the cluster state is stable and U unstable. We show two examples of computer experiments in Fig. 4. It can be seen from these examples that the cluster state ($\Phi=5\theta$) is stable [Fig. 4(a)] and the cluster state ($\Phi=6\theta$) is unstable [Fig. 4(b)] in the case of $(K,\tau)=(1,1)$. Table I shows that the stability of cluster states is modified by a time delay τ. These numerical results are verified theoretically in Sec. IV. In passing we note that in the cases $(K,\tau)=(1,0),(1,1),(1,2)$, Eq. (12) has only one solution Ω for any Φ. In the case of negative K, qualitatively similar results are obtained as to the linear stability of cluster states.

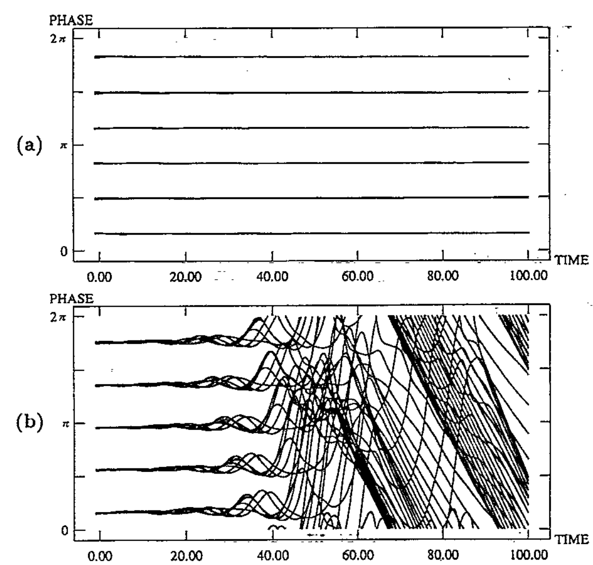

FIG. 4. Time evolution of stable and unstable cluster states. (a) $(K,\tau)=(1,1)$ and $\Phi=5\theta$ with $\Omega=0.684037$ from Eq. (12). The 6-cluster state ($\Phi=5\theta$) is stable. (b) $(K,\tau)=(1,1)$ and $\Phi=6\theta$ with $\Omega=0.782194$ from Eq. (12). The 5-cluster state ($\Phi=6\theta$) is unstable and finally attracted to the antiphase (30-cluster) state.

C. Relative stability among cluster states

In this section we study the relative stability among linearly stable cluster states. In the case $\tau\neq0$, we have no theory at the moment to investigate this important nonlinear phenomenon and we rely on computer simulations with random noise.

We consider the case in which Eq. (12) has more than one solution. To be concrete we study the 1-cluster state ($\Phi=0$), which is very stable for the attractive coupling (see Sec. III A) for, say, $(K,\tau)=(4,2)$. With this parameter Eq. (12) has five solutions Ω_1 $(=-2.58576)$, Ω_2 $(=-1.99362)$, Ω_3 $(=0.111940)$, Ω_4 $(=1.65276)$, and Ω_5 $(=2.89484)$ for $\Phi=0$. In order to investigate the stability of these solutions, we employ the following initial conditions:

$$\phi_k(t)=\Omega_0(t+\tau)+\alpha \quad (-\tau\le t\le 0) \tag{15}$$

for all k, where α is an arbitrary constant in the range $[0,2\pi]$ and Ω_0 is a parameter of our experiments. In Fig. 5 we show one typical trajectory of an experiment with $\Omega_0=-1.0$. In this case it is seen that the system is attracted to the 1-cluster state (Ω_3). It is to be noted that all the oscillators have the same phase through the transition. In the experiment with $\Omega_0=2.3$ the final state was a

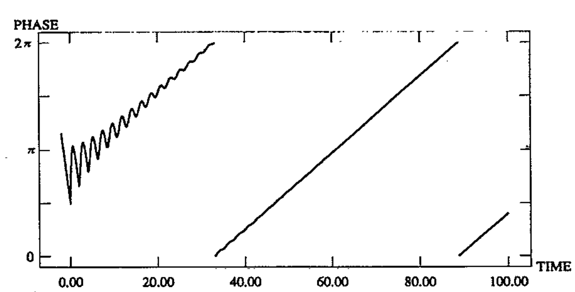

FIG. 5. Time evolution of phases $\phi_k(t)$ $(k=1,\ldots,30)$ with initial condition (15) $(\Omega_0=-1)$. The system is attracted to the 1-cluster state $(\Omega=\Omega_3)$.

1-cluster state (Ω_5). From other similar experiments and the theory given in Sec. IV, we find that the 1-cluster states with Ω_1, Ω_3, and Ω_5 are stable, and those with Ω_2 and Ω_4 are unstable.

Now the problem we pursue is the relative stability among these three stable 1-cluster states. In order to investigate this nonlinear problem we take the following Langevin equation:

$$\frac{d\phi_k(t)}{dt}=1+\frac{K}{2}\sum_{j=k\pm1}\sin\{\phi_j(t-\tau)-\phi_k(t)\}+\eta_k(t) \quad (k=1,\ldots,N)\ , \qquad (16)$$

where the random force $\eta_k(t)$ denotes a Gaussian white noise

$$\langle\eta_k(t)\rangle=0,\quad \langle\eta_k(t)\eta_j(t')\rangle=2D\delta_{kj}\delta(t-t')\ . \qquad (17)$$

As stated in Sec. II, a cluster state is uniquely defined by the phase difference Φ and the frequency Ω_b of Eq. (12). The result of our experiment is shown in Fig. 6. We set $D=1.0\times10^{-7}K$ and take $\Omega_0=\Omega_1$ in Eq. (15) for the initial condition. From Fig. 6 it can be seen that the 2-cluster state emerges as a transient state and then finally this state is attracted to the 1-cluster state (Ω_3). Similarly the 1-cluster state (Ω_5) is finally attracted to the 1-cluster state (Ω_3). On the contrary, the 1-cluster state (Ω_3) remains stable under the noise disturbance. From these experiments we conclude that the 1-cluster state (Ω_3) is the most stable.

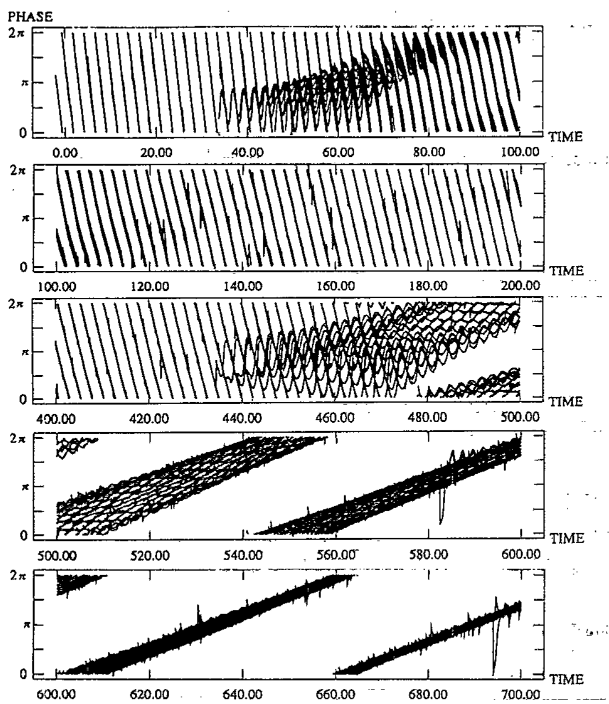

FIG. 6. Time evolutions of phases $\phi_k(t)$ ($k=1,\ldots,30$) under the noise for $(K,\tau)=(4,2)$, $D=1.0\times10^{-7}K$. Initial state is the 1-cluster state ($\Phi=0,\Omega=\Omega_1$). Through the transient 2-cluster state ($\Phi=\pi$) the system is finally attracted to the 1-cluster state. We note that trajectories from $t=200$ to 400 are omitted. In this period the system was in the 2-cluster state. The system was in the 1-cluster state after $t=700$ at least until $t=1000$ when the experiment was stopped.

IV. THEORETICAL ANALYSIS

A. Energy analysis for the system without a time delay

When there is no time delay, the energy of the system can be introduced to study properties of cluster states. In the case of $\tau=0$ we modify Eq. (6) with use of

$$\phi_k(t)=1t+\tilde{\phi}_k(t)\ . \qquad (18)$$

$\tilde{\phi}_k(t)$ denotes the phase in the coordinate rotating with frequency 1. Inserting Eq. (18) into Eq. (6) we obtain

$$\frac{d\phi_k}{dt}=\frac{K}{2}\sum_{j=k\pm1}\sin(\phi_j-\phi_k) \qquad (19)$$

$$=-\frac{\partial V(\phi_1,\ldots,\phi_N)}{\partial\phi_k}\ , \qquad (20)$$

where with the convention $\phi_{N+1}=\phi_1$,

$$V(\phi_1,\ldots,\phi_N)=-\frac{K}{2}\sum_{j=1}^{N}\cos(\phi_{j+1}-\phi_j) \qquad (21)$$

after omitting the tilde on $\phi_k(t)$. Therefore it follows that

$$\frac{dV}{dt}=\sum_k\frac{\partial V}{\partial\phi_k}\frac{d\phi_k}{dt}=-\sum_k\left[\frac{d\phi_k}{dt}\right]^2\le0\ . \qquad (22)$$

This means that the energy $V(\phi_1,\ldots,\phi_N)$ is the Lyapunov function for the system without a time delay. We also consider the energy of the cluster state which are introduced in Sec. II. From the definition of Φ by Eqs. (7) and (21), the energy per oscillator of the cluster state with phase Φ, $E_{\rm CS}$, is

$$E_{\rm CS}=\frac{V}{N}=-\frac{K}{2}\cos\Phi\ . \qquad (23)$$

Based on the energy Eq. (23), we can examine the linear stability of cluster states. Let us consider the energy of the state which is slightly perturbed from a cluster state (Φ). If the phase difference is

$$\phi_{k+1}-\phi_k=\Phi+\delta\phi_k\ , \qquad (24)$$

where $\delta\phi_k$ represents a small perturbation, the energy per oscillator of this state is given as

$$E=-\frac{K}{2N}\sum_{k=1}^{N}\cos(\Phi+\delta\phi_k)$$

$$=E_{\rm CS}+\frac{K}{2N}\left[\sum_{k=1}^{N}\delta\phi_k\right]\sin\Phi+\frac{K}{4N}\left[\sum_{k=1}^{N}(\delta\phi_k)^2\right]\cos\Phi\ . \qquad (25)$$

We note that due to the periodic boundary condition it holds that

$$\sum_{k=1}^{N} \delta\phi_k = 0 . \tag{26}$$

Thus

$$E = E_{CS} + \frac{K}{4N}\left[\sum_{k=1}^{N} (\delta\phi_k)^2\right]\cos\Phi . \tag{27}$$

This equation tells us that $\pi/2$ $(=\Phi_c)$ is the critical point. For attractive coupling $(K>0)$, if $\Phi<\Phi_c$, the perturbation makes the energy larger, which means that the cluster state (Φ) is linearly stable. The other case $(K<0)$ can be understood in the same way. This analysis explains the results of the case $(K,\tau)=(1,0)$ of Table I.

Next we consider the case in which the initial phases of oscillators are randomly distributed (see Sec. III A). Equation (23) tells us that in the case of attractive (repulsive) coupling, the energy E_{CS} increases (decreases) as Φ becomes large in the range of $0\le\Phi\le\pi$. The system starting from the random initial condition is likely to fall into the attractor with lower energy. Therefore it easily happens that the system with attractive (repulsive) coupling is attracted to the cluster state with smallest (largest) Φ. This explains the results in Sec. III A.

B. Linear stability analysis for systems with a time delay

In the cluster state the motion of oscillators is expressed by Eq. (11). We represent the phase of kth oscillator as

$$\phi_k(t) = \Omega t + k\Phi + \varepsilon a_k(t) , \tag{28}$$

where ε is assumed to be very small. Inserting Eq. (28) into Eq. (6), we compare terms with the same order of ε. The equation we get for the order ε^0 is Eq. (12). For the order ε^1, we obtain an equation of motion for $a_k(t)$ as

$$\begin{aligned}\frac{da_k(t)}{dt} &= \frac{K}{2}[\{a_{k-1}(t-\tau)-a_k(t)\}\cos(\Omega\tau+\Phi) \\ &\quad + \{a_{k+1}(t-\tau)-a_k(t)\}\cos(\Omega\tau-\Phi)] \\ &\equiv \alpha\{a_{k-1}(t-\tau)-a_k(t)\} \\ &\quad + \beta\{a_{k+1}(t-\tau)-a_k(t)\} ,\end{aligned} \tag{29}$$

where

$$a_1 = a_{N+1}, \quad a_0 = a_N \tag{30}$$

from the periodic boundary condition. We decompose $a_k(t)$ into Fourier modes as

$$a_k(t) = \sum_q f_q(t) e^{ikq} . \tag{31}$$

From Eqs. (30) and (31) we restrict q to

$$q = \frac{2\pi}{N} m \quad \left[-\frac{N}{2} < m \le \frac{N}{2}, \ m \text{ is an integer}\right] . \tag{32}$$

From Eqs. (29) and (31) we obtain the equation for $f_q(t)$ as

$$\frac{df_q(t)}{dt} = (\alpha e^{-iq}+\beta e^{iq}) f_q(t-\tau) - (\alpha+\beta) f_q(t) , \tag{33}$$

where

$$\alpha \equiv \frac{K}{2}\cos(\Omega\tau+\Phi), \quad \beta \equiv \frac{K}{2}\cos(\Omega\tau-\Phi) . \tag{34}$$

Denoting the Laplace transform of $f_q(t)$ by $\tilde{f}_q(s)$, we obtain from Eq. (33) the following equation:

$$\tilde{f}_q(s) = \frac{C}{s+(\alpha+\beta)-(\alpha e^{-iq}+\beta e^{iq})e^{-s\tau}} , \tag{35}$$

where C is the constant depending on the initial condition. Now we can examine the stability of the cluster state (Φ,Ω) as follows: First we calculate all the poles of $\tilde{f}_q(s)$ [Eq. (35)] for each mode q. If none of real parts of the poles for each q exceed zero, this cluster state is stable. If there exists a mode q_0, for which a pole has a positive real part, this cluster state is unstable. It is to be noted that Eq. (35) has a large number of poles because of the term $e^{-s\tau}$.

Examples of calculations are shown in Figs. 7 and 8. In Fig. 7 poles for $q=2\pi/5(m=6)$ and $\Phi=2\pi/5$ for $(K,\tau)=(1,1)$ are plotted. Real part of one of the poles exceed zero. Thus this cluster state is unstable. On the other hand, in Fig. 8 poles for all q and $\Phi=\pi/3$ for $(K,\tau)=(1,1)$ are plotted. The largest value of real parts of the poles is zero (Goldstone mode $q=0$). Thus this cluster state is stable. These calculations can explain the results of Table I and are not contradictory to Fig. 4.

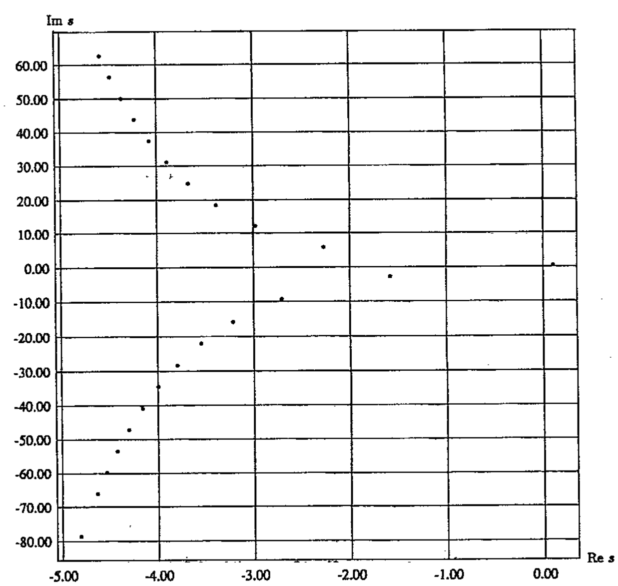

FIG. 7. Poles for $q=2\pi/5(m=6)$ and $\Phi=2\pi/5$ in the case of $(K,\tau)=(1,1)$.

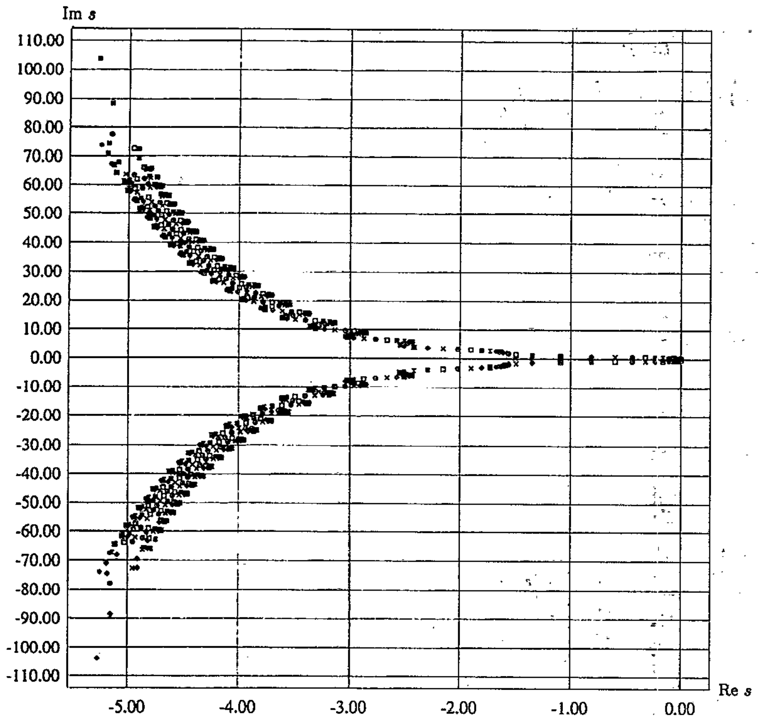

FIG. 8. Poles for all q and $\Phi=\pi/3$ in the case of $(K,\tau)=(1,1)$.

V. SUMMARY AND REMARKS

In this paper we mainly studied the oscillator systems with NN coupling with a time delay. The cluster state is found to be a rather general pattern of motion in NN coupling systems and can be classified by the phase difference Φ between neighboring oscillators. The role of the time delay consists in a modification of the collective frequency and of the linear stability of cluster states. In the case of no time delay, the results of the computer experiments on the linear stability of the cluster state and on the relative stability among cluster states are verified with theoretical analysis based on the energy of the system. In the case of a finite time delay, the linear stability can be analyzed with the technique of a Laplace transform, but we do not have a theory to discuss the relative (nonlinear) stability at present.

In the rest of this paper, we make two remarks (i) on the extension to a higher dimensional coupled oscillator system and (ii) on some computer experiments on the system with MF coupling.

(i) We can extend our one-dimensional NN coupling model to a higher dimensional one. For example, in the case of the two-dimensional oscillator system with NN coupling the equations of motion becomes

$$\frac{d\phi_{k_1k_2}(t)}{dt}=1-\frac{K}{4}\sum_{(i,j)}{}'\sin\{\phi_{ij}(t-\tau)-\phi_{k_1k_2}(t)\}$$
$$(k_1=1,\ldots,N_1;\; k_2=1,\ldots,N_2)\,, \quad (36)$$

where $\sum'_{(i,j)}$ means the summation over the nearest neighbors of the (k_1,k_2) oscillator and N_1 and N_2 are numbers of oscillators in the one axis and two axis, respectively. The cluster state can be expressed as

$$\phi_{k_1k_2}=\Omega t+k_1\Phi_1+k_2\Phi_2\,, \quad (37)$$

where Φ_1 and Φ_2 are phase differences between neighboring oscillators in the one axis and two axis, respectively. The collective frequency is calculated as

$$\Omega=1-\frac{K}{2}\sin(\Omega\tau)(\cos\Phi_1+\cos\Phi_2)\,. \quad (38)$$

Similarly in the case of n-dimensional systems, the equations of motion, the phase of the oscillators, and the collective frequency can be expressed as

$$\frac{d\phi_{k_1,\ldots,k_n}(t)}{dt}$$
$$=1-\frac{K}{2n}\sum_{(i_1,\ldots,i_n)}{}'\sin\{\phi_{i_1,\ldots,i_n}(t-\tau)-\phi_{k_1,\ldots,k_n}(t)\}$$
$$(k_1=1,\ldots,N_1;\ldots;k_n=1,\ldots,N_n)\,, \quad (39)$$

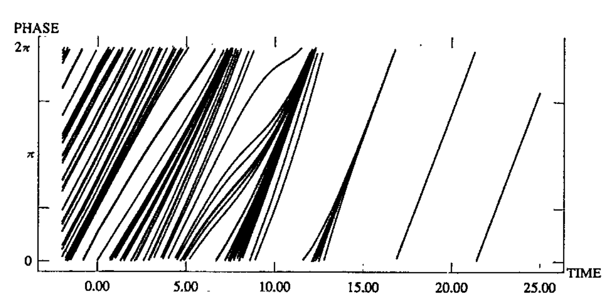

FIG. 9. Time evolution of oscillators $\phi_k(t)$ in the case of MF coupling for $(K,\tau)=(-1,2)$.

$$\phi_{k_1,\ldots,k_n}=\Omega t+k_1\Phi_1+\cdots+k_n\Phi_n\ , \tag{40}$$

$$\Omega=1-\frac{K}{n}\sin(\Omega\tau)(\cos\Phi_1+\cdots+\cos\Phi_n)\ . \tag{41}$$

(ii) In Sec. III B it was shown that the linear stability of the cluster state in the NN coupling system is modified by a time delay (see Table I). We also made the experiments on the system with MF coupling for the comparison. Sin(x) coupling was chosen as in Eq. (3) and a similar tendency as to the stability of the cluster state is acquired. We show a result of the case for the 1-cluster state. According to the result of Okuda [12], the 1-cluster state was not produced in the case of MF repulsive coupling with $\tau=0$. This means that the 1-cluster state is unstable. However, in the case $(K,\tau)=(-1,2)$ the 1-cluster state was produced as shown in Fig. 9, indicating that the time delay has modified the stability of the 1-cluster state.

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