Clustering behavior of time-delayed nearest-neighbor coupled oscillators

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We analyze both numerically and theoretically the dynamical behaviors of systems composed of
limit-cycle oscillators, which are coupled by a time-delayed nearest-neighbor interaction. In this system
we found cluster states which are characterized by a phase difference of neighboring oscillators and their
collective frequency. A time-delay turns out to have an important effect on the stability of various clus-
ter states.

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I. INTRODUCTION

Quite a few papers have been devoted to the dynamics of populations of coupled oscillators. Such populations
are known to model many systems in physics [1,2], biology [3–6], and chemistry [7]. It has been shown that cou-
pled limit-cycle oscillators with weak coupling can be
described by a system of phase-coupled oscillators, where
each oscillator is described by a single variable, its phase
[7]. Therefore, if we could show some features for a sys-
tem of phase coupled oscillators, we would be able to ex-
pect that the same is true for more general oscillator sys-
tems.

One of the remarkable phenomenon exhibited by a sys-
tem of coupled oscillators is synchronization or entrain-
ment; when oscillators, not necessarily identical to each
other, are allowed to mutually interact, it sometimes hap-
pens that the dynamics of oscillators are entrained to an
attractor in which every oscillator performs a synchron-
nized (frequency- or phase-locked) oscillation. It is report-
ed that when the natural frequency of each oscillator is
distributed around an average one, there exists a critical
strength of coupling, above which oscillators can be syn-
chronized [8–10].

Furthermore an attractor, which represents a so-called
cluster state, has been found for an oscillator system with
mean-field coupling, i.e., a globally coupled oscillator sys-
tem [11,12]. This state is characterized by the coexis-
tence of subgroups each of which consists of fully
phase-synchronized oscillators. This clustering phenom-
emon is interesting from the standpoint of a symmetry
breaking because a collection of dynamical elements is
spontaneously divided into some groups. In a sense the
entrained or phase-locked state, mentioned before, may
be called a one-cluster state. The opposite extremity is
the antiphase state [13,14] in which all the elements of
the system (the number is N) oscillate with the same fre-
quency and with equal spacing 2π/ in phase (a N-
cluster state). Thus a system can have various cluster
states.

In this paper we mainly investigate clustering behaviors in a phase-coupled oscillator system. The mod-
el systems may be variously classified even if we confine our discussion to phase-coupled systems. First, if we pay
attention to the range of interaction among oscillators, the models may belong to either the mean-field (MF)
[8,11,12] or short-range (SR) [9,10,15–18,20] interaction model. Due to the simplification of the analysis in the
case of MF coupling, the MF model has attracted much
attention. However, spatial structures, which are absent
in the MF model, are expected to yield a variety of self-
organized phenomena in a system with SR coupling.

This point of view we are mainly concerned with the
SR model. Second, we now turn to the time delay of
the interaction [19–21]. In models which are related to
physical and/or chemical phenomena, a time delay in the
interaction among oscillators does not play an important
role. However, in models which describe interaction
among neurons or some biological elements, it is some-
times important to take into account effects of finite time
required for information transmission between two ele-
ments. Therefore we consider both cases τ = 0 and τ ≠ 0.

Summarizing the above we study the collective behavior of the time-delayed nearest-neighbor (NN) cou-
pled oscillator systems. Such systems have been studied
by some authors. It is reported that due to the time delay
there exists a multitude of synchronized solutions [19]
and that time delay leads to a frequency depression [20].
However, it has not been reported whether or not cluster-
ing behavior can be observed in the NN coupling systems
and how the stability of cluster states is modified under
the influence of a finite time delay if such states exist.

This is the main topic of this paper.

This paper is organized as follows. In Sec. II we
present our model and in Sec. III results of our computer
experiments are reported with the main emphasis put on
the clustering behavior for the NN coupling systems. In
Sec. IV some theoretical studies are performed with use
of both an energy principle for the case τ = 0 and the
linear stability analysis for the case with a finite time de-
lay (τ ≠ 0). Section V contains some remarks on generali-
ization to higher dimensional systems and summary of the
results obtained in this paper.

II. MODEL

In this section we consider a one-dimensional system
composed of N phase-coupled oscillators whose dynamics

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are described as
\[
\frac{d\phi_k(t)}{dt} = \omega_0 + \frac{K}{I} \sum_j \sin[\phi_j(t-\tau) - \phi_k(t)] \tag{1}
\]
\[ (k=1, \ldots, N) , \]
where \( \phi_k(t) \) denotes the phase of the \( k \)th oscillator at time \( t \) and \( \omega_0 \) the common natural frequency, i.e., the frequency in the absence of interaction among oscillators. \( K \) is the coupling strength, \( I \) is the number of oscillators with which one oscillator interacts, and \( \tau \) is a time delay. In Sec. V we will comment on the extension to a higher dimensional model. Since we are mainly concerned with the NN coupling, the summation variable \( j \) takes the value \( k-1 \) and \( k+1 \), and \( I=2 \). We also consider, for the sake of comparison, the case of MF coupling for which \( j \) takes all the values except for \( k \) and \( I=N-1 \). We note that \( \phi_{N+1} \) is identical to \( \phi_1 \) due to the periodic boundary condition. Furthermore, by variable transformations
\[
\omega_0 t = \tilde{t}, \quad \frac{K}{\omega_0} = \tilde{K}, \quad \omega_0 \tau = \tilde{\tau}, \quad \phi_k(t) = \tilde{\phi}_k(\tilde{t}) \tag{2}
\]
we get the dimensionless equation
\[
\frac{d\tilde{\phi}_k(\tilde{t})}{d\tilde{t}} = 1 + \frac{K}{\tilde{K}} \sum_j \sin[\tilde{\phi}_j(\tilde{t}-\tilde{\tau}) - \tilde{\phi}_k(\tilde{t})] \tag{3}
\]
\[ (k=1, \ldots, N) . \]

In what follows, we consider Eq. (3) with tildes omitted for simplicity.

Systems with the MF coupling and no time delay (\( \tau = 0 \)) have been studied intensively. For \( \tau = 0 \) it is natural to term the coupling with \( K > 0 \) (\( K < 0 \)) as attractive (repulsive), because a small deviation \( \Theta_{jk} = \phi_j - \phi_k \) tends to decrease (increase) under the interaction \( K \sin \Theta_{jk} \). We use this terminology even in the case of \( \tau \). These points will be discussed in detail in Sec. IV when we study the stability of cluster solutions based on an energy analysis.

In anticipation of the various cluster states which are found in our computer simulation, and also for convenience of the latter discussion, we give here a systematical rule to produce cluster states. Before proceeding to the case of the NN coupling we first explain the (symmetric) cluster states for the case of the MF coupling which are discussed in detail by Okuda [12]. A symmetric \( N_c \)-cluster state, where \( N_c \) is one of the divisors of \( N \), is defined as the state in which each cluster \( l \) \( (l=0,1, \ldots, N_c-1) \), consisting of \( N/N_c \) oscillators with the same phase, rotates with the same frequency \( \Omega \). Thus, when the phase of \( N_c \) clusters are equally separated, the phase \( \phi_\ell \) of cluster \( \ell \) is described as
\[
\phi_\ell = \Omega t + \frac{2\pi \ell}{N_c} \quad (l=0, \ldots, N_c-1) . \tag{4}
\]
In the case of \( \tau = 0 \), \( \Omega \) is determined to be
\[
\Omega = 1 + \frac{K}{N_c} \sum_{\ell=0}^{n-1} \sin \left[ \frac{2\pi \ell}{N_c} \right] . \tag{5}
\]
Since there are many ways of distributing \( N \) oscillators into \( N_c \) groups with equal number of elements \( (=N/N_c) \), the solution (4) and (5) is highly degenerate. The case of \( N_c = N \) has attracted special attention because of the \((N-1)! \) attractors (a so-called attractor crowding) [13,14].

We now turn to the cluster state in the NN case. The model (3) is explicitly written as
\[
\frac{d\phi_k(t)}{dt} = 1 + \frac{K}{2} \sum_{j=k \pm 1} \sin[\phi_j(t-\tau) - \phi_k(t)] . \tag{6}
\]
In this case the phase difference
\[
\Phi = \phi_{k+1}(t) - \phi_k(t) , \tag{7}
\]
which turns out to be independent of \( k \) and \( t \), plays the fundamental role as \( N_c \) does in the case of MF model. Equation (7) means that, in constructing a cluster state characterized by \( \Phi \), we first specify the phase \( \phi_1 \) of the first oscillator. Then we take \( \phi_2 = \phi_1 + \Phi \) and so on. Since \( \Phi \) denotes a phase, it is in the range \( 0 \leq \Phi < 2\pi \). However, if \( \Phi \) happens to be larger than \( \pi \), \( \Phi = \phi_k - \phi_{k+1} \) can be taken in the range \( 0 \leq \Phi < \pi \) and we follow the prescription \( \phi_{N-c} = \phi_1 + \Phi \), \( \phi_{N-c-1} = \phi_{N-c} + \Phi \), and so on. Thus hereafter we will restrict \( \Phi \) in the range \( 0 \leq \Phi < \pi \). With use of phase unit
\[
\theta = \frac{2\pi}{N} , \tag{8}
\]
\( \Phi \) is expressed, from the condition \( \Phi = 2n\pi \), as
\[
\Phi = n\theta , \quad n = \begin{cases} 0, \ldots, N/2 & \text{for } N \text{ even} \\ 0, \ldots, (N-1)/2 & \text{for } N \text{ odd} \end{cases} . \tag{9}
\]
Thus, the number of the cluster \( N_c \) is determined, when \( \Phi \neq 0 \), from the condition
\[
\Phi N_c = n\theta N_c = 2\pi m , \tag{10}
\]
where \( m \) denotes the smallest integer that satisfies Eq. (10). For example, in the case of \( N = 30 \), \( N_c = 10 \) for \( n = 3 \), \( N_c = 5 \) for \( n = 12 \), and so on. For \( \Phi = 0 \), it is evident that \( N_c = 1 \).

As to the dynamics of the cluster states, the phase of \( k \)th oscillator evolves in time as
\[
\phi_k(t) = \Omega t + k\Phi \quad (k=1, \ldots, N) , \tag{11}
\]
where \( \Omega \) is the common frequency of the oscillators. Inserting Eq. (11) into Eq. (6), we get
\[
\Omega = 1 - K\sin(\Omega t)\cos\Phi . \tag{12}
\]
Even if \( \Phi \) (or \( n \)) is given, depending on the parameter \( K \) and \( \tau \), the transcendental equation (12) for \( \Omega \) may have more than one solution, which we denote as \( \Omega_b \) \( (b = 1, \ldots, B) \). Thus in order to unambiguously define a
cluster state we must specify not only $\Phi$, Eq. (7), but also $\Omega_k$, and we denote the cluster state by $(\Phi, \Omega_k)$.

III. COMPUTER EXPERIMENTS FOR NN COUPLING

We performed three types of computer experiments to examine (i) the structure of cluster states, (ii) the linear stability of the cluster states, and (iii) the relative nonlinear stability among the linearly stable cluster states. Computer experiments were performed by integrating Eq. (6) with the use of the Runge-Kutta-Gill algorithm with the time step $\Delta t = 0.01$. We have checked that decreasing the time step does not affect the results. We also examined systems consisting of different number of oscillators. Since general features of clustering behavior does not depend on $N$, we show results for $N = 30$ in the following. The initial conditions of the oscillators were chosen randomly or some cluster states according to the purpose of experiments.

A. Examples of cluster states

In this subsection we show some examples of clustering behavior of the system with $N = 30$, thus phase unit $\theta = \pi/15$ [see Eq. (8)]. For the initial condition we set

$$\phi_k(t) = 1(t + \tau) + \alpha_k \quad (\tau \leq t \leq 0),$$

(13)

where phases $\alpha_k \ (k = 1, \ldots, N)$ are randomly distributed over $[0, 2\pi]$. For the attractive coupling ($K > 0$) either a 1-cluster state $(\Phi = 0)$ or a 30-cluster state $(\Phi = \pi)$ is produced depending on the initial condition and for the repulsive coupling ($K < 0$) cluster states with large $\Phi$, either a 15-cluster state $(\Phi = 14\theta)$ or a 2-cluster state $(\Phi = 15\theta)$ is produced. Some examples of clustering behaviors are given in Fig. 1, which depicts trajectories of time evolution of oscillators $\phi_k(t) \ (k = 1, \ldots, N)$. These cluster states are also produced in the case of finite time delay, as shown in Fig. 2, $(K, \tau) = (1, 1)$. As mentioned in Sec. III B, other linear stable cluster solutions can exist in the case studied above. However, we could not get such cluster states in our simulations. This suggests that the 1- or 30-cluster states in the case of attractive coupling and 2- or 15-cluster states in the case of repulsive coupling have the largest basin of attraction. When there is no time delay these results are physically explainable from an energy principle (Sec. IV). We also examine the time evolution of the average frequency $\langle \phi \rangle = (1/N)\sum_k \phi_k(t)$ for various time delays $\tau$ when a 1-cluster state is produced under the initial condition (13). It can be seen from Fig. 3 that the averaged frequency of clusters becomes small as the time delay $\tau$ becomes large. In the case of parameters studied in Fig. 3, Eq. (12) has only one solution, which is indicated by symbols in Fig. 3.

FIG. 1. Time evolution of oscillators $\phi_k(t) - \Omega t \ (k = 1, \ldots, 30)$ in the case of no time delay. (a) 1-cluster state $(\Phi = 0)$ for $(K, \tau) = (1, 0)$ with $\Omega = 1.0$ from Eq. (12). (b) 2-cluster state $(\Phi = \pi)$ for $(K, \tau) = (-1, 0)$ with $\Omega = 1.0$ from Eq. (12).

FIG. 2. Time evolution of oscillators in the case of a finite time delay for $(K, \tau) = (1, 1)$ with $\Omega = 0.510973$ from Eq. (12). The 1-cluster state $(\Phi = 0)$ is produced. The vertical axis denotes the number of oscillators. The dark region indicates that the phase takes a value between 0 and $\pi/4$.

FIG. 3. Time evolution of averaged frequencies $(1/N)\sum_k \phi_k(t)$ for various time delays (from the top $\tau = 0.0, 0.2, 0.4, 0.6, 0.8, 1.0$ $(K = 1)$. Increasing a time delay induces frequency depression. The symbols indicate the solutions of Eq. (12) $\bullet \ (\tau = 0.0); \bigstar, \ 0.833977 \ (\tau = 0.2); \times, \ 0.717084$ $(\tau = 0.4); \circ, \ 0.630602 \ (\tau = 0.6); \bigtriangleup, \ 0.563973 \ (\tau = 0.8); \hat{\bigtriangleup}, \ 0.510973 \ (\tau = 1.0)$, which precisely agree with the experiments.
TABLE I. Summary of computer experiments for linear stability with $K=1$. Here the phase unit $\theta=\pi/15$ and $N_c$ denotes the number of clusters. $S$ means the cluster is stable and $U$ unstable.

<table>
<thead>
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<th>$\Phi$</th>
<th>$N_c$</th>
<th>$\tau$</th>
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<th>2</th>
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<td>$S$</td>
<td>$S$</td>
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<tr>
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<td>$S$</td>
<td>$S$</td>
<td>$S$</td>
<td>$S$</td>
</tr>
<tr>
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<td>15</td>
<td>$S$</td>
<td>$S$</td>
<td>$S$</td>
<td>$U$</td>
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<td>$S$</td>
<td>$S$</td>
<td>$U$</td>
</tr>
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<td>$S$</td>
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<td>$S$</td>
<td>$U$</td>
</tr>
<tr>
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<td>$S$</td>
<td>$U$</td>
<td>$U$</td>
</tr>
<tr>
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<td>$U$</td>
<td>$U$</td>
<td>$U$</td>
</tr>
<tr>
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<td>30</td>
<td>$U$</td>
<td>$U$</td>
<td>$U$</td>
<td>$U$</td>
</tr>
<tr>
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<td>$U$</td>
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<td>$S$</td>
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</tr>
</tbody>
</table>

B. Linear stability of the cluster state

Next we study linear stability of various cluster states. For this purpose the initial condition is set as

$$\phi_k(t) = \Omega(t+\tau) + k\Phi + \varepsilon_k \quad (\tau \leq t \leq 0),$$

where $\varepsilon_k$ ($k=1, \ldots, N$) are random numbers in the range $[-\pi/100, \pi/100]$. This initial state is the one that is slightly perturbed from a cluster state ($\Phi$). If this cluster state is restored in spite of a perturbation, then it is stable; otherwise this cluster state is unstable. Our simulation results are summarized in Table I. In this table $S$ denotes that the cluster state is stable and $U$ unstable. We show two examples of computer experiments in Fig. 4. It can be seen from these examples that the cluster state ($\Phi=50\theta$) is stable [Fig. 4(a)] and the cluster state ($\Phi=60\theta$) is unstable [Fig. 4(b)] in the case of $(K, \tau)=(1,1)$. Table I shows that the stability of cluster states is modified by a time delay $\tau$. These numerical results are verified theoretically in Sec. IV. In passing we note that in the cases $(K, \tau)=(1,0), (1,1), (1,2)$, Eq. (12) has only one solution $\Omega$ for any $\Phi$. In the case of negative $K$, qualitatively similar results are obtained as to the linear stability of cluster states.

C. Relative stability among cluster states

In this section we study the relative stability among linearly stable cluster states. In the case $\tau \neq 0$, we have no theory at the moment to investigate this important nonlinear phenomenon and we rely on computer simulations with random noise.

We consider the case in which Eq. (12) has more than one solution. To be concrete we study the 1-cluster state ($\Phi=0$), which is very stable for the attractive coupling (see Sec. III A) for, say, $(K, \tau)=(4,2)$. With this parameter Eq. (12) has five solutions $\Omega_1 \ldots \Omega_5$, where $\Omega_1 = -2.58576$, $\Omega_2 = -1.99362$, $\Omega_3 = 0.111940$, $\Omega_4 = 1.65276$, and $\Omega_5 = 2.89484$ for $\Phi=0$. In order to investigate the stability of these solutions, we employ the following initial conditions:

$$\phi_k(t) = \Omega_0(t+\tau) + \alpha \quad (\tau \leq t \leq 0)$$

for all $k$, where $\alpha$ is an arbitrary constant in the range $[0,2\pi]$ and $\Omega_0$ is a parameter of our experiments. In Fig. 5 we show a typical trajectory of an experiment with $\Omega_0 = -1.0$. In this case it is seen that the system is attracted to the 1-cluster state ($\Omega_3$). It is to be noted that all the oscillators have the same phase through the transition. In the experiment with $\Omega_0 = 2.3$ the final state was a
1-cluster state ($\Omega_1$). From other similar experiments and
the theory given in Sec. IV, we find that the 1-cluster
states with $\Omega_1$, $\Omega_2$, and $\Omega_3$ are stable, and those with $\Omega_2$
and $\Omega_4$ are unstable.

Now the problem we pursue is the relative stability
among these three stable 1-cluster states. In order to
investigate this nonlinear problem we take the following
Langevin equation:

$$\frac{d\phi_k(t)}{dt} = 1 + \frac{K}{2} \sum_{j=k \pm 1} \sin[\phi_j(t - \tau) - \phi_k(t)] + \eta_k(t)$$

$(k = 1, \ldots, N)$,  \hspace{1cm} (16)

where the random force $\eta_k(t)$ denotes a Gaussian white
noise

$$\langle \eta_k(t) \rangle = 0, \quad \langle \eta_k(t) \eta_j(t') \rangle = 2D \delta_{kj} \delta(t - t')$$ \hspace{1cm} (17)

As stated in Sec. II, a cluster state is uniquely defined by
the phase difference $\Phi$ and the frequency $\Omega_k$ of Eq. (12).
The result of our experiment is shown in Fig. 6. We set
$D = 1.0 \times 10^{-7}K$ and take $\Omega_2 = \Omega_1$ in Eq. (15) for the
initial condition. From Fig. 6 it can be seen that the 2-
cluster state emerges as a transient state and then finally
this state is attracted to the 1-cluster state ($\Omega_2$). Similarly
the 1-cluster state ($\Omega_3$) is finally attracted to the 1-
cluster state ($\Omega_3$). On the contrary, the 1-cluster state
($\Omega_4$) remains stable under the noise disturbance. From
these experiments we conclude that the 1-cluster state
($\Omega_3$) is the most stable.

IV. THEORETICAL ANALYSIS

A. Energy analysis for the system without a time delay

When there is no time delay, the energy of the system
can be introduced to study properties of cluster states. In
the case of $\tau = 0$ we modify Eq. (6) with use of

$$\phi_k(t) = t + \tilde{\phi}_k(t)$$ \hspace{1cm} (18)

$\tilde{\phi}_k(t)$ denotes the phase in the coordinate rotating with
frequency 1. Inserting Eq. (18) into Eq. (6) we obtain

$$\frac{d\phi_k}{dt} = \frac{K}{2} \sum_{j=k \pm 1} \sin(\phi_j - \phi_k)$$ \hspace{1cm} (19)

$$= - \frac{\partial V(\phi_1, \ldots, \phi_N)}{\partial \phi_k}$$ \hspace{1cm} (20)

where with the convention $\phi_{N+1} = \phi_1$,

$$V(\phi_1, \ldots, \phi_N) = - \frac{K}{2} \sum_{j=1}^{N} \cos(\phi_{j+1} - \phi_j)$$ \hspace{1cm} (21)

after omitting the tilde on $\phi_k(t)$. Therefore it follows that

$$\frac{dV}{dt} = \sum_k \frac{\partial V}{\partial \phi_k} \frac{d\phi_k}{dt} = - \sum_k \left[ \frac{d\phi_k}{dt} \right]^2 \leq 0$$ \hspace{1cm} (22)

This means that the energy $V(\phi_1, \ldots, \phi_N)$ is the
Lyapunov function for the system without a time delay.
We also consider the energy of the cluster state which are
introduced in Sec. II. From the definition of $\Phi$ by Eqs.
(7) and (21), the energy per oscillator of the cluster state
with phase $\Phi$, $E_{CS}$, is

$$E_{CS} = \frac{V}{N} = - \frac{K}{2} \cos \Phi$$ \hspace{1cm} (23)

Based on the energy Eq. (23), we can examine the
linear stability of cluster states. Let us consider the energy
of the state which is slightly perturbed from a cluster
state ($\Phi$). If the phase difference is

$$\phi_{k+1} - \phi_k = \Phi + \delta \phi_k$$ \hspace{1cm} (24)

where $\delta \phi_k$ represents a small perturbation, the energy per
oscillator of this state is given as

$$E = - \frac{K}{2N} \sum_{k=1}^{N} \cos(\Phi + \delta \phi_k)$$

$$= E_{CS} + \frac{K}{2N} \left[ \sum_{k=1}^{N} \delta \phi_k \right] \sin \Phi + \frac{K}{4N} \left[ \sum_{k=1}^{N} (\delta \phi_k)^2 \right] \cos \Phi$$ \hspace{1cm} (25)

We note that due to the periodic boundary condition it
holds that
\[
\sum_{k=1}^{N} \delta \phi_k = 0 .
\]

Thus
\[
E = E_{CS} + \frac{K}{4N} \left( \sum_{k=1}^{N} (\delta \phi_k)^2 \right) \cos \Phi .
\]

This equation tells us that \( \pi/2 \) (\( = \Phi_2 \)) is the critical point. For attractive coupling (\( K > 0 \)), if \( \Phi < \Phi_2 \), the perturbation makes the energy larger, which means that the cluster state (\( \Phi \)) is linearly stable. The other case (\( K < 0 \)) can be understood in the same way. This analysis explains the results of the case (\( K, \tau = (1,0) \)) of Table I.

Next we consider the case in which the initial phases of oscillators are randomly distributed (see Sec. III A). Equation (23) tells us that in the case of attractive (repulsive) coupling, the energy \( E_{CS} \) increases (decreases) as \( \Phi \) becomes large in the range of \( 0 \leq \Phi \leq \pi \). The system starting from the random initial condition is likely to fall into the attractor with lower energy. Therefore it easily happens that the system with attractive (repulsive) coupling is attracted to the cluster state with smallest (largest) \( \Phi \). This explains the results in Sec. III A.

### B. Linear stability analysis for systems with a time delay

In the cluster state the motion of oscillators is expressed by Eq. (11). We represent the phase of \( k \)th oscillator as
\[
\phi_k(t) = \Omega t + k \Phi + \varepsilon a_k(t) ,
\]
where \( \varepsilon \) is assumed to be very small. Inserting Eq. (28) into Eq. (6), we compare terms with the same order of \( \varepsilon \). The equation we get for the order \( \varepsilon^0 \) is Eq. (12). For the order \( \varepsilon^1 \), we obtain an equation of motion for \( a_k(t) \) as
\[
\frac{da_k(t)}{dt} = \frac{K}{2} \left[ \{ a_{k-1}(t-\tau) - a_k(t) \} \cos (\Omega \tau + \Phi) \\
- \{ a_{k+1}(t-\tau) - a_k(t) \} \cos (\Omega \tau - \Phi) \right] \\
\equiv \alpha \{ a_{k-1}(t-\tau) - a_k(t) \} \\
+ \beta \{ a_{k+1}(t-\tau) - a_k(t) \} ,
\]
where
\[
a_1 = a_{N+1}, \quad a_0 = a_N
\]
from the periodic boundary condition. We decompose \( a_k(t) \) into Fourier modes as
\[
a_k(t) = \sum_q f_q(t) e^{i q \theta} .
\]
From Eqs. (30) and (31) we restrict \( q \) to
\[
q = \frac{2\pi}{N} m \left\{ -\frac{N}{2} < m \leq \frac{N}{2} , \quad m \text{ is an integer} \right\} .
\]
From Eqs. (29) and (31) we obtain the equation for \( f_q(t) \) as
\[
\frac{df_q(t)}{dt} = (ae^{-i\eta} + be^{i\eta}) f_q(t-\tau)^{-i(\alpha + \beta)} f_q(t) ,
\]
where
\[
\alpha = \frac{K}{2} \cos (\Omega \tau + \Phi), \quad \beta = \frac{K}{2} \cos (\Omega \tau - \Phi) .
\]
Denoting the Laplace transform of \( f_q(t) \) by \( \tilde{f}_q(s) \), we obtain from Eq. (33) the following equation:
\[
\tilde{f}_q(s) = \frac{C}{s + (\alpha + \beta) - (ae^{-i\eta} + be^{i\eta}) e^{-\tau s}} ,
\]
where \( C \) is the constant depending on the initial condition. Now we can examine the stability of the cluster state \( (\Phi, \Omega) \) as follows: First we calculate all the poles of \( \tilde{f}_q(a) \) [Eq. (35)] for each mode \( q \). If none of real parts of the poles for each \( q \) exceed zero, this cluster state is stable. If there exists a mode \( q_0 \), for which a pole has a positive real part, this cluster state is unstable. It is to be noted that Eq. (35) has a large number of poles because of the term \( e^{-\tau s} \).

Examples of calculations are shown in Figs. 7 and 8. In Fig. 7 poles for \( q = 2\pi/5 (m = 6) \) and \( \Phi = 2\pi/3 \) for \( (K, \tau) = (1,1) \) are plotted. Real part of one of the poles exceed zero. Thus this cluster state is unstable. On the other hand, in Fig. 8 poles for all \( q \) and \( \Phi = \pi/3 \) for \( (K, \tau) = (1,1) \) are plotted. The largest value of real parts of the poles is zero (Goldstone mode \( q = 0 \)). Thus this cluster state is stable. These calculations can explain the results of Table I and are not contradictory to Fig. 4.

### FIG. 7. Poles for \( q = 2\pi/5 (m = 6) \) and \( \Phi = 2\pi/3 \) in the case of \( (K, \tau) = (1,1) \).
V. SUMMARY AND REMARKS

In this paper we mainly studied the oscillator systems with NN coupling with a time delay. The cluster state is found to be a rather general pattern of motion in NN coupling systems and can be classified by the phase difference \( \Phi \) between neighboring oscillators. The role of the time delay consists in a modification of the collective frequency and of the linear stability of cluster states. In the case of no time delay, the results of the computer experiments on the linear stability of the cluster state and on the relative stability among cluster states are verified with theoretical analysis based on the energy of the system. In the case of a finite time delay, the linear stability can be analyzed with the technique of a Laplace transform, but we do not have a theory to discuss the relative (nonlinear) stability at present.

In the rest of this paper, we make two remarks (i) on the extension to a higher dimensional coupled oscillator system and (ii) on some computer experiments on the system with MF coupling.

(i) We can extend our one-dimensional NN coupling model to a higher dimensional one. For example, in the case of the two-dimensional oscillator system with NN coupling the equations of motion becomes

\[
\frac{d\phi_{k_1,k_2}(t)}{dt} = 1 - \frac{K}{4} \sum_{(i,j)} \sin[\phi_{ij}(t-\tau) - \phi_{k_1,k_2}(t)]
\]

\[\phi_{k_1,k_2} = \Omega t + k_1\Phi_1 + k_2\Phi_2,\]  

(37)

where \( \Phi_1 \) and \( \Phi_2 \) are phase differences between neighboring oscillators in the one axis and two axis, respectively. The collective frequency is calculated as

\[
\Omega = 1 - \frac{K}{2} \sin(\Omega \tau)(\cos\Phi_1 + \cos\Phi_2).
\]

(38)

Similarly in the case of \( n \)-dimensional systems, the equations of motion, the phase of the oscillators, and the collective frequency can be expressed as

\[
\frac{d\phi_{k_1,\ldots,k_n}(t)}{dt} = 1 - \frac{K}{2n} \sum_{(i_1,\ldots,i_n)} \sin[\phi_{i_1,\ldots,i_n}(t-\tau) - \phi_{k_1,\ldots,k_n}(t)]
\]

\[\phi_{k_1,\ldots,k_n} = \Omega t + k_1\Phi_1 + \ldots + k_n\Phi_n,\]  

(39)

where \( \sum_{(i,j)} \) means the summation over the nearest neighbors of the \((k_1,k_2)\) oscillator and \( N_1 \) and \( N_2 \) are numbers of oscillators in the one axis and two axis, respectively. The cluster state can be expressed as

\[
\phi_{k_1,\ldots,k_n}(t)
\]

\[
= \Omega t + k_1\Phi_1 + \ldots + k_n\Phi_n,
\]

(36)

FIG. 8. Poles for all \( q \) and \( \Phi = \pi/3 \) in the case of \( (K,\tau) = (1,1) \).

FIG. 9. Time evolution of oscillators \( \phi_q(t) \) in the case of MF coupling for \( (K,\tau) = (-1,2) \).
\[ \phi_{k_1, \ldots, k_n} = \Omega_t + k_1 \Phi_1 + \cdots + k_n \Phi_n, \]

\[ \Omega = 1 - \frac{K}{n} \sin(\Omega \tau) (\cos \Phi_1 + \cdots + \cos \Phi_n). \]

(ii) In Sec. III B it was shown that the linear stability of the cluster state in the NN coupling system is modified by a time delay (see Table I). We also made the experiments on the system with MF coupling for the comparison. \( \sin(x) \) coupling was chosen as in Eq. (3) and a similar tendency as to the stability of the cluster state is acquired. We show a result of the case for the 1-cluster state. According to the result of Okuda [12], the 1-cluster state was not produced in the case of MF repulsive coupling with \( \tau = 0 \). This means that the 1-cluster state is unstable. However, in the case \( (K, \tau) = (-1, 2) \) the 1-cluster state was produced as shown in Fig. 9, indicating that the time delay has modified the stability of the 1-cluster state.

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